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HIGHER STRUCTURES FROM SIMPLE LATTICE MODELS

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Abstract

Recent developments have revealed that global symmetries of physical systems can be described and reconstructed from topological defects of various codimensions, which constrain the dynamics of the theory [1]. These defects can be locally deformed and fused according to the discrete combinatorial data of a structure known as a fusion category—without affecting the underlying physics. This realization has sparked a surge of new results and unprecedented interdisciplinary collaboration across mathematics [2, 3], high-energy theory [4, 5], and condensed matter physics [6, 7, 8].

Structures and symmetries that were once seen as niche or exotic are now being understood as part of a broader unifying mathematical framework. While category theory and higher structures often lie outside the standard training of physicists —and many of the first examples of categorical symmetries are very unfamiliar to mathematicians —these modern methods emerge naturally in some of the simplest physical models, and the underlying mathematics is surprisingly intuitive.

In this thesis, we explore how the language of topological defects and fusion categories arises directly from studying the simplest and most fundamental classical lattice model: the 2D Ising model. Beginning with the physics of the early nineteenth century, we develop from the ground up the theories of classical and quantum statistical mechanics, duality, gauge symmetry, conformal field theory, and non-invertible symmetries. My goal is to demonstrate how topological and categorical structures are not merely abstract and esoteric mathematical formalisms, but are in fact necessary to study even the most familiar and ubiquitous physical theories.

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Acknowledgements

To my first love, Alexis. Thank you for encouraging and allowing me to fall for my second love, mathematical physics.

I am also deeply indebted to my mentor, Adrian Ocneanu. While this thesis is entirely my own creation, I am forever grateful that you welcomed me into your beautiful world of mathematics and physics. The intuitive and visual approach to mathematical physics that I have learned from you now guides not only my work, but also how I perceive the world on a fundamental level. Thank you for helping me to rewire my brain and gain the confidence to see connections between things that I never thought possible, and for teaching me to construct my own unique understanding of even the most basic concepts.

Chapter 1

Introduction

Given a discrete collection of sites that each host a magnetic dipole moment of atomic spins pointing either up $(+1)$ or down (-1), how do macroscopic properties like ferromagnetism arise in materials? This is the question that was posed by physicist Wilhelm Lenz to his PhD student Ernst Ising in 1920. With the benefit of hindsight, this question was incredibly cruel to ask to a young PhD student; nevertheless, in his 1924 PhD thesis [9], Ising ultimately introduced and analytically solved a statistical mechanical system on a 1 dimensional lattice – the aptly named Ising model – which provides a partial answer to Lenz's question. Unknowingly, Ising's work opened a pandora's box of rich mathematical structures in topology, category theory, representation theory, and topological/conformal field theories that arise naturally from his namesake model and its classical and quantum cousins. Similar in spirit to [10], my intention is for this thesis to serve as a stepping stone between the physics and mathematics perspectives on lattice models. Allowing for such dialogue between condensed matter physics and pure mathematics is crucial to the joint developments of both fields.

The single most fundamental problem looming over physicists of the 21st century is the so-called "strong coupling problem." That is, to rigorously understand the behavior of quantum many body systems and quantum field theories whose interactions are too strongly coupled that they lie beyond the usual low-energy approximation regime of perturbation theory. Beyond just demonstrating an inaptitude of mathematical rigor, strongly coupled quantum systems —in both the discrete and field-theoretic context —display exotic behavior which frequently force physicists to reconsider all of their existing models of nature. This chaotic atmosphere creates a breeding ground for brand new mathematical structures as the line between physics and mathematics becomes increasingly blurred.

For instance, while physicists had largely accepted Landau theory's theory of symmetry breaking as providing the classification and characterization of all possible phases of matter and the transitions between them, a flurry of experimental discoveries in the 1980's, beginning with the fractional quantum hall (FQH) effect, demonstrated how this theory breaks down in strongly coupled systems. Specifically, the FQH exhibited states which, despite being in different phases, obeyed the same symmetries. Not only this, but FQH excitations

had charges that were equal to fractions of the charge of the electron (previously thought to be the smallest unit of charge) and had exchange statistics that were neither bosonic nor fermionic.

Discoveries such as FQH, high T_c superconductors, chiral spin liquids, and Witten's construction of the Chern-Simons topological quantum field theory from the Jones Polynomial motivated Xiao-Gang Wen to coin the term **topological order**. This refers to states that have properties including, but not necessarily limited to, topological ground state degeneracy, anyonic excitations, long range entanglement, and a robustness against low-energy perturbations. These new phases of matter could not be described by local order parameters or spontaneous symmetry breaking, and thus fell outside the scope of Landau theory. Instead, their defining features were intrinsically non-local and topological in nature. To characterize such phases, physicists turned to new mathematical structures that could encode non-trivial braiding and fusion of excitations, ground state degeneracy on nontrivial manifolds, and the topological interactions of defects.

This led naturally to the study of **topological defects**, extended objects that can be inserted into a quantum field theory without changing local observables, but which carry global data about the topological phase. These defects —lines, surfaces, or higher-dimensional generalizations —can be manipulated, deformed, and fused in ways that obey intricate algebraic relations. Remarkably, these fusion and braiding rules are not arbitrary: they are governed by the axioms of a (higher) **fusion category**, a mathematical structure that formalizes how objects (like anyons or defect lines) combine and interact.

Fusion categories provide a unifying language for describing the algebra of anyonic excitations in topologically ordered systems, and more broadly, for organizing the behavior of topological defects in a wide class of quantum field theories. In particular, they encode not only the fusion rules of excitations but also the internal symmetries of a topological phase —including so-called *categorical symmetries* that generalize ordinary group symmetries. Thus, the study of topological order, born from physical puzzles like the FQHE, has ultimately led to a deeper understanding of quantum phases through the lens of higher algebra, category theory, and topological field theory.

On the high-energy side, the landscape is more speculative. It has long been recognized that our current formulation of quantum field theory is mathematically incomplete. The central object of QFT —the path integral —is often ill-defined and typically requires perturbative expansions to extract any meaningful physical predictions. This limitation becomes particularly stark in the study of strongly coupled systems, such as quantum chromodynamics, which lie beyond the reach of perturbative methods.

In the absence of transformative experimental breakthroughs in high-energy physics, a promising direction is to leverage the new mathematical structures that have proven successful in condensed matter contexts—especially those related to topological order—to deepen our understanding of quantum field theories in the non-perturbative regime. In fact, high-energy physicists are no strangers to topological defects and fusion categories; some of the earliest examples arose in the study of rational conformal field theories, where defect lines and modular tensor categories play a central role in encoding operator algebras and dualities.

Many of the most promising approaches to quantum gravity also make essential use of these structures. For instance, the Ponzano–Regge and Turaev–Viro-Ocneanu models successfully discretize 3D gravity by triangulating spacetime and associating objects and morphisms from tensor categories to the vertices, edges, faces, and tetrahedra of the triangulation [11, 12]. Similarly, string theory makes extensive use of categorical structures by evolving a 2D worldsheet —governed by conformal field theory —through higher-dimensional spacetimes in order to echo the role of topological defects and tensor categories in capturing interactions and symmetry.

In order to truly capture the essence of the beauty of conformal field theory and generalize it to more complicated models, we need to develop higher categorical mathematics with which we can fill spacetime itself with algebraic computations through networks of topological defects of various codimensions.

Our job as theoretical physicists is not to take all that has come before us as given and hope that it supports us as we venture into speculation. Rather, it is to make use of our greatest privilege —hindsight —to reinterpret past formalisms in ways that allow for new theories to come directly to us. The giants whose shoulders we stand on could only see so high; they could not construct their theories with the explicit purpose of supporting those yet to come.

If we want to build another floor atop an existing building, we cannot simply start construction on the roof. We must first reassess the infrastructure of the entire building through the lens of our evolving needs —tearing parts down, reinforcing others, and laying new support beams —before we can even think about building upward. If the preparation is done correctly, though, we may find ourselves not just able to build another floor, but many more, reaching as high as we are able to see.

This is the philosophy behind this thesis. Nothing groundbreaking will be presented in isolation, but rather I hope that my deliberate choices to collect this information, new and old alike, into a single document unified under a single modern viewpoint will be groundbreaking in its own right. My intention is not for this thesis to be a testament to or celebration of the things I have done, but rather a preparation for and a statement of what I intend to do.

We begin in Chapter 2 by reformulating the framework of classical statistical mechanics from an information-theoretic point of view according to the school of thought popularized by E.T Jaynes in the 50's [13]. We will then visit the 2d classical Ising model with a particular focus on its symmetries, like Kramers-Wannier duality, and the transfer matrix formalism. From the transfer matrix formalism, we rigorously derive the connection between the 2d classical Ising model and the 1d quantum Ising model in a transverse field by taking an extreme anisotropic limit as the vertical lattice spacing goes to zero and identifying the vertical lattice direction with Wick rotated time.

In Chapter 3, we study the nuances of the 1d quantum Ising model in great detail. We begin by studying the implications of the global $\mathbb{Z}_2$ symmetry of the model and its spontaneous breaking, from which we observe how this symmetry can be reconstructed from topological defects living on the links of the chain. From here, we see how Kramers-Wannier duality of the 1d quantum Ising model can be obtained directly from gauging the global $\mathbb{Z}_2$ symme-

try through a process in which we sum over all ways of inserting defects on the links and then impose a Gauss law on each site. Next, we introduce the Jordan-Wigner and Bogoliubov Transformations which allow us to map the Ising chain to a model of non-interacting fermions. As in the previous chapter, we take a continuum limit —this time at the critical point —to obtain the free Majorana fermion conformal field theory. We close by briefly touching on the connections of rational conformal field theories with topological defects and fusion categories without diving too far into the formalism.

We conclude in Chapter 4 by motivating and introducing the formalism of category theory as a necessity to understand topological defects in generality. We first define categories and basic notions like functors, being careful to avoid abstract machinery like the Yoneda Lemma, while keeping our eye toward physically motivated constructions. Next, we define an additive structure on categories, culminating in the definition of a 2-vector space. We then define a monoidal product and an intuitive graphical calculus, on top of which we will define a pivotal structure and finally give the definition of a fusion category. After exploring some of the rich structures that fusion categories carry, we conclude by seeing fusion rules in action through an exploration of topological defects in the 2d classical Ising model.

Chapter 2

The 2D Classical Ising Model

While the majority of this thesis is dedicated to presenting the categorical and topological structures that arise from studying lattice models, it is vital that we build up a common language from the ground up to show how the importance of these structures follows logically from studying even the simplest physical models. As such, we will begin by rigorously defining and solving the simplest classical lattice model from which nearly all modern tools used by theoretical physics can be derived from. Beyond the fact that the methods and symmetries utilized in analyzing the classical Ising model are important to understanding more complex theories, we will show that this model naturally gives rise to mathematically sophisticated notions like topological defects, gauge theory, conformal field theory, and more.

2.1 Crash Course in Classical Statistical Mechanics

Statistical mechanics is the branch of physics concerned with describing how macroscopic phenomena—like pressure and temperature—emerge from systems of unfathomably many interacting degrees of freedom. It is often a favorite of mathematicians, as it is not a "branch of physics" in the usual sense. Rather than beginning with observed laws of nature and seeking a general framework to explain them, statistical physics provides a powerful toolbox that can be applied to virtually any large, complex system—physical or otherwise.

道生一，一生二，二生三，三生万物。

The above quote is taken from the Tao Te Ching, and roughly translates to: 'The Tao gives birth to one, one gives birth to two, two gives birth to three, and three gives birth to ten thousand things.' One possible interpretation is that from the unknowable (the Tao) arises the concept of individual entities. From the existence of one comes the potential for interaction; from a few, a system; and from the system, the emergence of collective behavior greater than the sum of its parts. This is the motivating philosophy behind statistical mechanics—well captured by the modern phrase: "More is different." A single molecule of air cannot have pressure, yet no one would deny that air pressure is crucial to our macroscopic world. Many fundamental physical concepts are, in fact, emergent.

Statistical mechanics distinguishes between microstates and macrostates. A **microstate** specifies all the relevant data (e.g., position and momentum) of each individual particle in a system. Since such systems typically involve on the order of 10^{23} particles, knowledge of the exact microstate is essentially impossible. A **macrostate**, on the other hand, encodes the configuration of the entire system in terms of coarse-grained, measurable quantities like temperature, pressure, or magnetization.

Rather than relying on physical axioms to relate microstates and macrostates, we can instead frame this as an information-theoretic problem [14]. Given what the macroscopic information we can directly observe from a system, what can we infer about the distribution of microstates? The best we can hope for is a probability distribution over all microstates that (1) matches the observed macroscopic data in expectation, and (2) minimizes uncertainty. This naturally leads to the principle of maximum entropy.

Definition 2.1.1: Classical Statistical Mechanical System

A **classical statistical mechanical system** is a triple (X, μ, H), where X is a set of microstates, μ is a measure on X, and $H : X \to \mathbb{R}_{\geq 0}$ is a real-valued function called the **Hamiltonian**. For a microstate $x \in X$, the value $H(x)$ is interpreted as its **energy**.

We assume that X follows a probability distribution $p : X \to \mathbb{R}_{\geq 0}$ such that

$$\int_X p(x)\, d\mu(x) = 1 \tag{2.1}$$

In classical information theory, the information gained from observing x is $-\log p(x)$, called the **self-information** of x.

Definition 2.1.2: Gibbs Entropy

Given a system (X, μ, H) with probability distribution p, the **Gibbs entropy** is defined as:

$$\sigma := -k \int_X p(x) \log p(x)\, d\mu(x) \tag{2.2}$$

where $k \approx 1.38 \times 10^{-23}$ J/K is Boltzmann's constant.

The Gibbs entropy simply tells us the expected amount of self-information contained in a microstate. Typically, statistical mechanics refers to entropy as the "disorder of a system," but in this context, a better explanation is that entropy is a measure of our ignorance of which microstate actually describes our system. It is clear then that the most logical choices, or at the least the safest choices for the probability distribution of X are those which maximize the Gibbs entropy. This is known as **the principle of maximum entropy**.

Suppose we know that the measured value of the macroscopic energy of the system is $E \in \mathbb{R}_{\geq 0}$. Given what we know, the best possible probability distribution $p(x)$ is the one

which maximizes the Gibbs Entropy subject to the constraint

$$\langle H \rangle := \int_X H(x)p(x)d\mu(x) = E \tag{2.3}$$

We can find a solution to this optimization problem by introducing a Lagrange Multiplier β, from which we find the solution

$$p(x) = \frac{\exp\left(-\beta H(x)\right)}{Z(\beta)}, \quad \text{where} \quad Z(\beta) = \int_X \exp\left(-\beta H(x)\right)d\mu(x) \tag{2.4}$$

Definition 2.1.3: The Boltzmann Distribution

We say that a statistical mechanical system (X, μ, H) with probability distribution p is in **Thermal Equillibrium** if p maximizes the Gibbs Entropy subject to a given energy constraint. In other words, if the microstates follow the probability distribution:

$$p(x) = \frac{\exp\left(-\beta H(x)\right)}{Z(\beta)} \tag{2.5}$$

which is known as **The Boltzmann Distribution**. We refer to $Z(\beta)$ as the **Partition Function** of the system.

We often write $\beta = \frac{1}{kT}$, where T is the temperature of the system, where k appears in the expression to ensure that $\beta H(x)$ is dimensionless. The motivation for writing β in terms of temperature is that as $T \to \infty$ (that is, $\beta \to 0$), all microstates become equally probable, which matches our physical environment.

Although it appears simply as a normalization factor, the partition function $Z(\beta)$ is deeply fundamental: nearly all macroscopic observables can be derived from it.

Fact 2.1.4

For a system in thermal equilibrium, the expected energy is given by:

$$E = \langle H \rangle = -\frac{d}{d\beta} \log Z(\beta). \tag{2.6}$$

Proof. By direct calculation:

$$\begin{aligned} -\frac{d}{d\beta} \log Z(\beta) &= -\frac{1}{Z(\beta)} \frac{d}{d\beta} Z(\beta) \\ &= \frac{1}{Z(\beta)} \int_X H(x) \exp(-\beta H(x))\, d\mu(x) \\ &= \int_X H(x)p(x)\, d\mu(x) \\ &= \langle H \rangle \end{aligned} \tag{2.7}$$

□

Fact 2.1.5

The Gibbs entropy is related to the partition function via:

$$\sigma = k\left(\log Z(\beta) - \beta\frac{d}{d\beta}\log Z(\beta)\right). \tag{2.8}$$

Proof. Using the definition of $p(x)$, we find:

$$\begin{aligned}\sigma &= -k\int_X p(x)\log p(x)\,d\mu \\ &= -k\int_X p(x)\log\left(\frac{\exp(-\beta H(x))}{Z(\beta)}\right)d\mu.\end{aligned} \tag{2.9}$$

This becomes:

$$\sigma = k\left(\beta\int_X H(x)p(x)\,d\mu + \log Z(\beta)\right) = k(\beta E + \log Z(\beta)). \tag{2.10}$$

Substituting the expression for E gives the desired result. □

Definition 2.1.6: Free Energy

The **free energy** of a system is defined as the quantity:

$$F := E - T\sigma = -kT\log Z(\beta). \tag{2.11}$$

Observe that $T\sigma$ has units of energy and reflects the uncertainty of the system due to entropy. So the free energy $F = E - T\sigma$ subtracts the "inaccessible" portion of the total energy. We interpret F as the energy that is actually available for doing work. Moreover, thermal equilibrium is equivalent to the minimization of free energy, reinforcing this interpretation.

Since $F = -kT\log Z$, the partition function $Z(\beta)$ effectively counts the number of **accessible microstates** at a given temperature. And because $Z = \int_X \exp(-\beta H(x))\,d\mu$, we interpret $\exp(-\beta H(x))$ as a measure of how accessible a given microstate is. This matches our physical intuition: at high temperatures, more states become accessible.

With this, we have built a logically coherent and mathematically precise foundation for classical statistical mechanics. We now turn to one of the simplest and most instructive models in this framework: the classical Ising model.

2.2 Introducing The Classical Ising Model

Consider a **finite** 2d square lattice with N sites, where each site i is labelled by a discrete variable $\sigma_i \in \{-1, 1\}$. We refer to each σ_i as a **spin** and a particular assignment of $+1$'s and -1's to each site as a **spin configuration**, $[\sigma]$. Visualizing $+1$'s as upward pointing

arrows and -1's as downward pointing arrows, a particular lattice configuration can be seen in Figure 1.

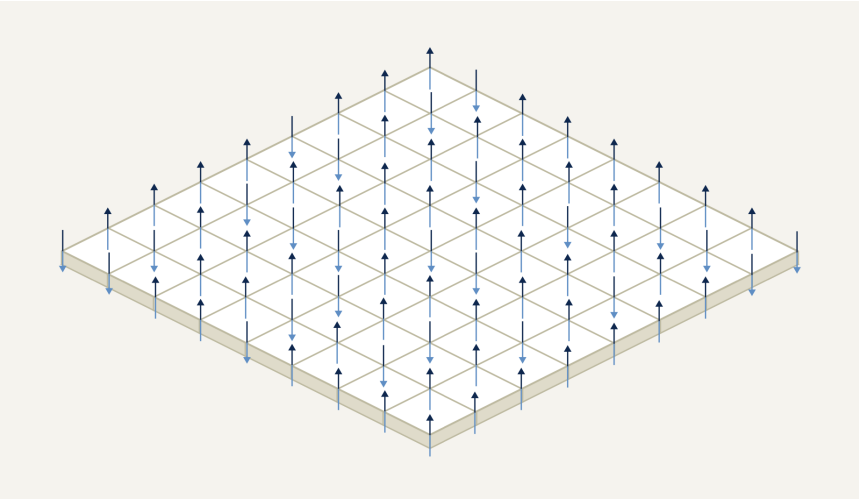


Figure 2.1: A particular configuration of the 2D Classical Ising Model [15]

We would then like to define a Hamiltonian on our set of configurations. For every pair of neighboring sites $\langle ij \rangle$, we define an interaction term given by $-J\sigma_i\sigma_j$, where $J \in \mathbb{R}$ is a coupling constant which we will later tune to obtain different behaviors of our system. Then, for every individual site i, there will be a contribution to the energy given by $-h\sigma_i$, where $h \in \mathbb{R}$ is a coupling constant which is interpreted as the strength of an external transverse magnetic field. The Hamiltonian is then simply the sum over all of these contributions.

Definition 2.2.1: Ising Model

The 2D classical Ising Model in a magnetic field of strength h is a classical statistical mechanical system whose microstates are spin configurations $[\sigma]$ and whose Hamiltonian is given by:

$$H([\sigma], h) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

We specify that the Hamiltonian is a function of h to emphasize that the magnetic field strength is something we control in order to probe the physics of the model in its natural state. Thus, the quantities we really care about will be those which are taken in the limit as $h \to 0$. Accordingly, we will denote the partition function function at zero magnetic field as simply: $Z(\beta) := Z(\beta, h = 0)$. Equipped with our Hamiltonian, the partition function for the 2d classical Ising model is:

$$Z(\beta, h) = \sum_{[\sigma]} \exp\left(-\beta H([\sigma], h)\right) \tag{2.12}$$

where we are summing over all possible spin configurations $[\sigma]$. The free energy **per site** of

the theory is then given by:

$$f(\beta, h) = -\frac{kT}{N} \log Z(\beta, h) \tag{2.13}$$

. From here, we can in principle compute any macroscopic quantity we could possibly need to understand the large scale behavior of the Ising Model. Perhaps most importantly is the **Magnetization**, $M = \langle \sigma_j \rangle$, where the expectation value $\langle \sigma_j \rangle$ is computed using the Boltzmann Distribution at zero magnetic field ($h = 0$). The translation invariance of this theory tells us that this value will be the same no matter which site j we choose, allowing us to refer to any one of these expectation values unambiguously as M.

Fact 2.2.2

The magnetization can be found from the free energy per site via: $M = \lim_{h \to 0} -\frac{\partial f(\beta,h)}{\partial h}$

Proof. Recalling the definition of the Boltzmann Distribution, observe that $M = \langle \sigma_j \rangle$. Then, since we are summing over all possible configurations, the expected value of one spin is the same as the expected value of the average of all N spins; i.e

$$\begin{aligned}
M &= \left\langle \frac{1}{N} \sum_i \sigma_i \right\rangle \\
&= \frac{1}{NZ(\beta)} \sum_{[\sigma]} \sum_k \sigma_k \exp\left(J\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right) \\
&= \lim_{h \to 0} \frac{1}{NZ(\beta, h)} \sum_{[\sigma]} \frac{1}{\beta} \frac{\partial}{\partial h} \exp\left(J\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j + h\beta \sum_k \sigma_k \right) \\
&= \lim_{h \to 0} \frac{kT}{N} \frac{1}{Z(\beta, h)} \frac{\partial Z(\beta, h)}{\partial h} \\
&= \frac{kT}{N} \frac{\partial}{\partial h} \log Z(\beta, h) \\
&= -\frac{\partial f(\beta, h)}{\partial h}
\end{aligned} \tag{2.14}$$

□

Another fundamental quantity that we care about is the connected correlation of two spins σ_i and σ_j: $\langle \sigma_i \sigma_j \rangle_c = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$. This quantity roughly tells us the mutual statistical dependence of the two spins. Since the Ising model is translation invariance, this dependence should not depend on the specific sites i and j, but rather just the distance between the sites $|i - j|$. We will see how the correlation depends on distance in a future section.

2.3 Kramers-Wannier Duality

Consider the 2d Classical Ising Model in zero magnetic field ($h = 0$). It is obvious that this theory has a global $\mathbb{Z}_2$ symmetry given by flipping all of the spins; i.e the Hamiltonian is

invariant under $\sigma_i \mapsto -\sigma_i \ \forall\, i$, as:

$$H([\sigma]) = \sum_{\langle ij \rangle} \sigma_i \sigma_j \mapsto H([-\sigma]) = \sum_{\langle ij \rangle} (-\sigma_i)(-\sigma_j) = H([\sigma]) \tag{2.15}$$

However, this is not the only symmetry of this model; there is in fact a very non-obvious duality in this theory, discovered by Kramers and Wannier in 1941 [16], relating the 2d Ising model at one temperature to another 2d Ising model at a different temperature. While discovered over 80 years ago, Kramers-Wannier duality has had a resurgence of interest in the last few years as people have realized that it is a part of a broader phenomenon known as non-invertible symmetries. We will discuss the modern perspective of such symmetries in later chapters, but it is important for us to first understand the historical context of Kramers and Wannier's observation.

Observe that the partition function for the 2d Ising Model in zero magnetic field can be written as:

$$\begin{aligned} Z(K) &= \sum_{[\sigma]} \exp\left(K \sum_{\langle ij \rangle} \sigma_i \sigma_j\right) \\ &= \sum_{[\sigma]} \prod_{\langle ij \rangle} \exp(K \sigma_i \sigma_j) \\ &= \sum_{[\sigma]} \prod_{\langle ij \rangle} \sum_{k=0}^{1} C_k(K) (\sigma_i \sigma_j)^k \end{aligned} \tag{2.16}$$

where $K := J\beta$, $C_0(K) = \cosh(K)$, and $C_1(K) = \sinh(K)$. In doing this, we have essentially introduced a new dynamical variable, $k = 0,\, 1$ on every link; we make this more explicit by writing the variable as k_l for each link l. After some algebraic manipulations of regrouping the spins and exchanging sums and products, we can write the partition function as:

$$Z(K) = \sum_{[\sigma]} \sum_{[k]} \prod_{l} C_{k_l}(K) \prod_{i} \sigma_i^{\sum_j k_{\langle ij \rangle}} \tag{2.17}$$

where performing a sum over $[k]$ denotes summing over all possible assignments of 0's and 1's to each of the k_l's, and $\sum_j k_{\langle ij \rangle}$ denotes summing over all k_l where l is a link anchored at a site i. Now, with all of the σ_i in a single term, we can perform the sum over $[\sigma]$ to obtain:

$$Z(K) = \sum_{[k]} \prod_{l} C_{k_l}(K) \prod_{i} 2\delta\left(\sum_{j} k_{\langle ij \rangle} \bmod 2\right) \tag{2.18}$$

where $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ otherwise. When the partition function is written this way, it is clear that a large portion of the terms in the sum over $[k]$ will have vanishing contributions. In fact, only those configurations in which **every vertex** has an **even number of edges** l incident to it with $k_l = 1$ will have a non-zero contribution.

If we visualize the links as being filled in with green if $k_l = +1$ and empty if $k_l = 0$, as in Figure 2.2, then enforcing that only configurations with an even number of $+1$ links attached to each vertex amounts to summing over configurations in which the links with $+1$ links only form closed loops! This notion is deeply related to gauge theory as we will see in the

next chapter; in fact, summing over configurations in which the k_l variables sum to an even number at each vertex is mathematically equivalent to summing over flat connections of $\mathbb{Z}_2$ gauge fields.

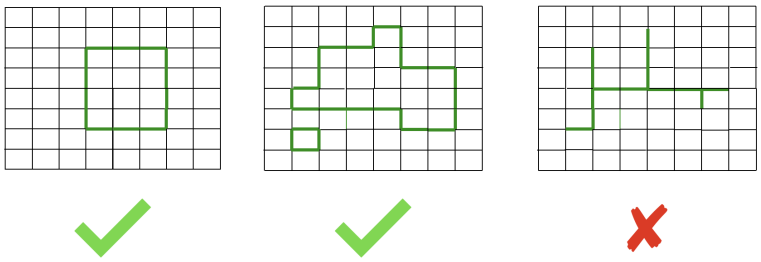


Figure 2.2: Configurations of link variables that contribute to the partition function

Because of this, we want to look for a new way of phrasing the k_l variables that remove this redundancy from the partition function. Kramers and Wannier found that the right way of doing so is to start by defining **the dual lattice**, whose vertices are the centers of the plaquettes (faces) of the original lattice, as seen in Figure 2.3.

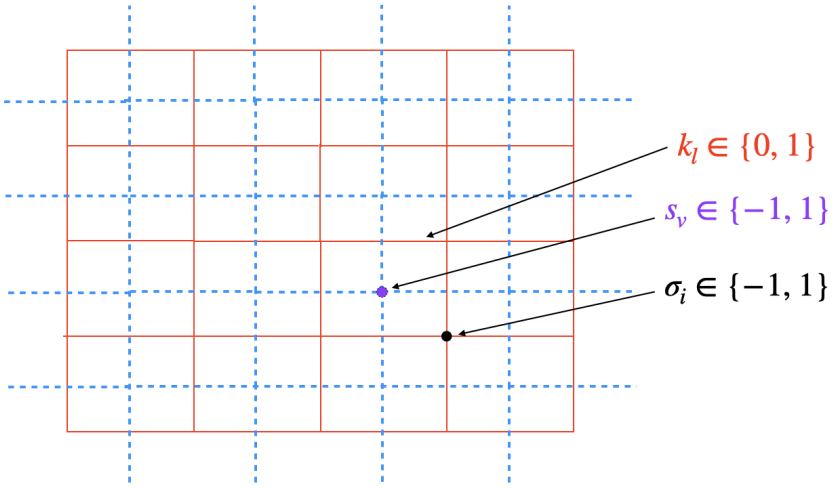


Figure 2.3: The original lattice (in red) along with its dual lattice (in blue).

On each vertex v of the dual lattice, we define new variables $s_v \in \{-1, 1\}$, denoting a particular assignment of values to these variables as $[v]$. Observe that every link l can be specified by two vertices on the dual lattice; namely, the centers of the two plaquettes that l is the interface between.

Claim: A New Expression for the Link Variables

The configurations $[k]$ which have non-vanishing contributions to the partition function are exactly those in which every k_l can be written as $k_l = \frac{1}{2}(1 - s_v s_w)$, where v and w are the two vertices of the dual lattice which specify l.

Proof. Suppose that $k_l = \frac{1}{2}(1 - s_v s_w)$ for every link l, as described above. Consider an arbitrary vertex of the original lattice i with edges labeled as in Figure 2.4. We want to show that $k_{(12)} + k_{(23)} + k_{(34)} + k_{(41)} = 2 - \frac{1}{2}(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1)$ is even for all possible configurations $[s]$ and that appropriate values of s_1, s_2, s_3, and s_4 can be chosen to make the sum equal to $0, 2$, and 4. Thus, let us check all possible cases for this vertex.

1. $\mathbf{s_1 = s_2 = s_3 = s_4 = +1}$: If only one edge is labeled -1, we have $k_{(12)} + k_{(23)} + k_{(34)} + k_{(41)} = 2 - \frac{1}{2}(1 + 1 + 1 + 1) = 0$
2. $\mathbf{s_1 = -1}$, $\mathbf{s_2 = s_3 = s_4 = +1}$: In this case $\sum k_l = 2 - \frac{1}{2}(-1 + 1 + 1 - 1) = 2$
3. $\mathbf{s_1 = s_2 = -1}$, $\mathbf{s_3 = s_4 = +1}$: If exactly two adjacent edges are labeled -1, we have $\sum k_l = 2 - \frac{1}{2}(1 - 1 + 1 - 1) = 2$
4. $\mathbf{s_1 = s_3 = -1}$, $\mathbf{s_2 = s_4 = +1}$: If exactly two opposite edges are labeled with -1, we have have $\sum k_l = 2 - \frac{1}{2}(-1 - 1 - 1 - 1) = 4$
5. $\mathbf{s_1 = s_2 = s_3 = -1}$, $\mathbf{s_4 = +1}$: If exactly three edges are labeled -1, we have $\sum k_l = 2 - \frac{1}{2}(+1 + 1 - 1 - 1) = 2$
6. $\mathbf{s_1 = s_2 = s_3 = s_4 = -1}$: Here, we have $\sum k_l = 2 - \frac{1}{2}(1 + 1 + 1 + 1) = 0$

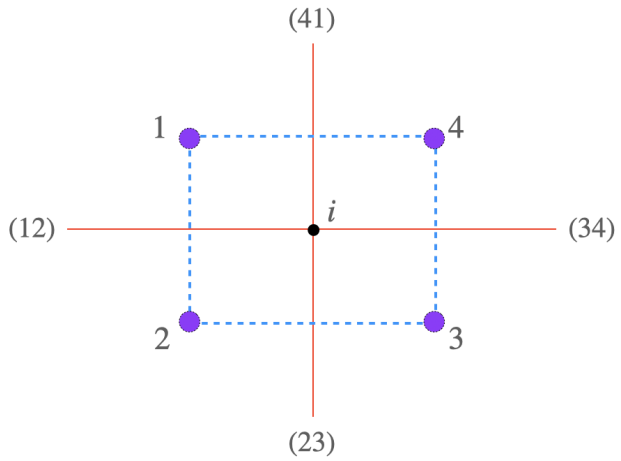


Figure 2.4: Arbitrary vertex i with four surrounded plaquettes labeled 1, 2, 3, 4.

Thus, every possible configuration $[s]$ ensures that the sum of all link variables k_l incident at each vertex is even. Also, a configuration can always be chosen to make this sum any desired even number from 0 to 4, which proves our claim. □

Now, some quick combinatorics tells us that there are exactly 2 configurations of $s_1, \cdots, s_4$ that give a sum of 0, 12 configurations that give a sum of 2, and 2 configurations that gives a sum of 4 at a given vertex. On the other hand, there is 1 configuration of $k_1, \cdots, k_4$ that gives a sum of zero at a vertex, $\binom{4}{2} = 6$ configurations of $k_1, \cdots, k_4$ that give a sum of 2, and 1 configuration of $k_1, \cdots, k_4$ that gives a sum of 4. Thus, if we replace k_l with our new expression in our partition function, we must include an extra factor of $\frac{1}{2}$ in order to account for the double counting if we sum over all configurations of $[s]$ on the dual lattice.

Now, performing the product over all sites i on the original lattice with our new definition of k_l that has no vanishing contributions to the partition function, we have:

$$Z(K) = \frac{1}{2}2^N \sum_{[s]} \prod_{(vw)} C_{\frac{1}{2}(1-s_v s_w)}(K) = 2^{N-1} \sum_{[s]} \prod_{(vw)} C_{k_{(vw)}}(K) \tag{2.19}$$

where (vw) refers to the link specified by the vertices v and w on the dual lattice. Lastly, notice that since $k_{(vw)} = 0, 1$, we can write:

$$\begin{aligned} C_{k_{(vw)}}(K) &= \cosh(K)\left(1 + k_{(vw)}(\tanh(K) - 1)\right) \\ &= \cosh(K) \exp\left(k_{(vw)} \log(\tanh(K))\right) \\ &= \cosh(K) \left(\tanh(K)\right)^{1/2} \exp\left(-\tfrac{1}{2} \log(\tanh(K))\, s_v s_w\right) \\ &= \sqrt{\cosh(K)\sinh(K)} \exp\left(-\tfrac{1}{2} \log(\tanh(K))\, s_v s_w\right) \end{aligned} \tag{2.20}$$

Thus, the partition function simplifies as:

$$\begin{aligned} Z(K) &= 2^{N-1} \sum_{[s]} \prod_{(vw)} \sqrt{\cosh(K)\sinh(K)} \exp\left(-\frac{1}{2} \log\left(\tanh(K)\right) s_v s_w\right) \\ &= \frac{1}{2}(\sinh(2K))^N \sum_{[s]} \exp\left(-\frac{1}{2} \log\left(\tanh\left(K\right)\right) \sum_{(vw)} s_v s_w\right) \end{aligned} \tag{2.21}$$

where the last equality follows from the fact that there are $2N$ edges (v, w) on our lattice with N vertices, as well as the identity $\sinh(2\beta) = 2\sinh(\beta)\cosh(\beta)$. But observe that the RHS is directly proportional to the partition function of another 2d Ising Model just at a different temperature! This is the amazing discovery that Kramers and Wannier realized in 1941; we can sum it up in one equation

$$Z_{\text{ Ising}}(K) = \frac{1}{2}(\sinh(2\beta))^N Z_{\text{Ising}}(\tilde{K}) \tag{2.22}$$

where

$$\tilde{K} = -\frac{1}{2} \log\left(\tanh\left(K\right)\right) \tag{2.23}$$

is referred to as the **dual coupling**. We have plotted the dual coupling as a function of the original coupling below in 2.5. From this, we can see that Kramers-Wannier duality is a mapping from a weakly coupled Ising Model to a strongly coupled Ising Model. Even better perhaps, thinking of the coupling J as fixed, we can instead think of this mapping as a self-duality between a given Ising model at a high temperature with itself at a low temperature.

Despite coming from seemingly unmotivated algebraic manipulations, this self-duality is an incredibly powerful and fundamental symmetry. It is typically much easier for us to understand the behavior of statistical or field theoretic models in the weakly coupled regime, in which we can Taylor expand the partition function. However, equipped with this duality, we can actually map the weak coupling expansion into the strongly coupled regime to understand the low temperature behavior perturbatively as well.

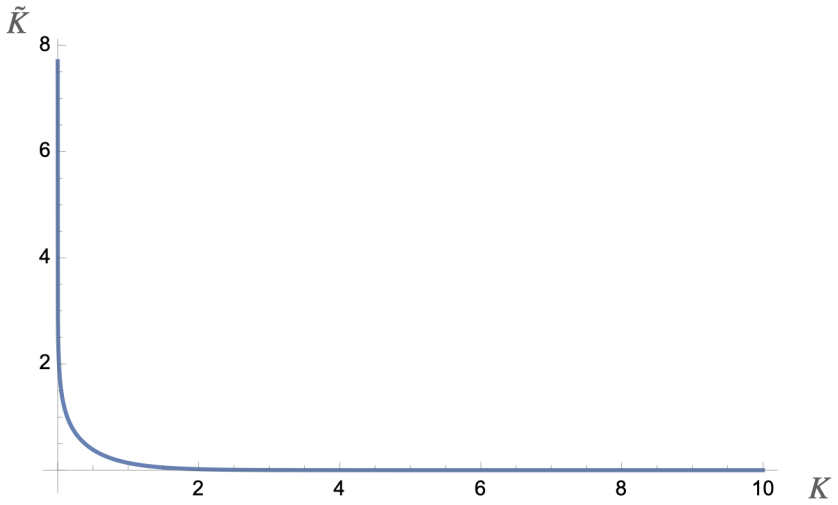


Figure 2.5: Dual coupling plotted against original coupling

Even this does not even capture the sheer strength and beauty of this duality. In fact, even if we considered our model to have a different coupling $K_{\langle ij\rangle}$ for every pair of spins on the lattice. This follows from the fact that the coupling J only manifests in Eq. 2.19 through the variable $C_{k_{(vw)}}(K)$ which is defined on links. Then, since the entire argument leading up to the dual partition function in Eq. 2.29 only depends on the local values of the variables on the links, we find each value of the coupling gets mapped to a dual coupling under Kramers-Wannier duality via:

$$\tilde{K}_{\langle ij\rangle} = -\frac{1}{2}\log\tanh K_{\langle ij\rangle} \tag{2.24}$$

This is an almost miraculously elegant aspect of the 2d Ising model that is often taken for granted.

Back to the case of uniform coupling, we can see that Kramers-Wannier Duality puts the following constraint on the free energy per site of the 2d Ising Model:

$$f(K) = \log(\sinh(2K)) + f(\tilde{K}) \tag{2.25}$$

This means that up to the addition of smooth analytic function of K, the expression for the free energy is invariant under Kramers-Wannier duality. Hence, if there is a single value K_c of the coupling at which the free energy or any of its derivatives have singularities – indicating a *phase transition* – it must occur at the *self-dual* point when $K_c = \tilde{K}_c$. From Eq. 2.23, it is easily found that the critical value for the coupling is:

$$K_c = \frac{1}{2}\log\left(1+\sqrt{2}\right) \tag{2.26}$$

While this relies on the assumption that there was indeed a single point in which the free energy or one of its derivatives was singular, an exact solution of the 2d Ising model by Onsager in 1941 later confirmed this assumption, showcasing the power of Kramers-Wannier

duality. Later in this thesis, we will find a closed-form expression for the free energy of the 2d Ising model in the thermodynamic limit using a different method than Onsager. We will revisit Kramers-Wannier duality twice more during this thesis, recasting it and the global $\mathbb{Z}_2$ symmetry of the 2d Ising Model in the framework of topological defects, showing the fundamental nature of these two symmetries and how they interact with one another.

2.4 Introducing the Transfer Matrix: A Path Integral Suitable for Mathematicians

In their original paper, Kramers and Wannier also found a convenient way of expressing the partition function $Z(\beta)$ simply as the trace of a suitable $2^n \times 2^n$ matrix (for an $m \times n$ square lattice), known as the transfer matrix. Unbeknownst to the physicists of the 40's, it turns out that the transfer matrix formalism is crucial to understanding how classical statistical models can be related to quantum field theories.

At this point we will begin referring to sites on the lattice by the coordinates (i, j), and their associated spin by $\sigma_{i,j}$. For convenience, we denote a configuration of a given row k as $\sigma^{(k)} := \{\sigma_{1,k}, \cdots, \sigma_{n,k}\}$. We will also impose periodic boundary conditions $\sigma_{ij} = \sigma_{i+n,j}$ and $\sigma_{ij} = \sigma_{i,j+m}$, effectively putting our model on a torus. Then, a given row has the following contribution to the energy of a given configuration:

$$E[\sigma^{(k)}] = -J \sum_{i=1}^{n} \sigma_{ik}\sigma_{i+1,k} \tag{2.27}$$

Furthermore, there is a contribution to the energy given by the interaction between two neighboring rows:

$$E[\sigma^{(k)}, \sigma^{(k+1)}] = -J \sum_{i=1}^{n} \sigma_{ik}\sigma_{i,k+1} \tag{2.28}$$

Now, for a given row k, we define a formal 2^n dimensional vector space V_k with an orthonormal basis given by the collection of all row configurations: $\{|\sigma^{(k)}\rangle\}_{[\sigma]}$, where we will henceforth be using bra-ket notation. As the rows are uniform in length, each of these vector spaces are the same and we will refer to each vector space unambiguously as V.

Then, we define the so-called **transfer matrix**, T, which takes in two neighboring row configurations and spits out their joint contribution to the partition function. It is defined by the following matrix elements:

$$\langle \sigma^{(k+1)} | \, T \, | \sigma^{(k)} \rangle = \exp -\beta(E[\sigma^{(k)}, \sigma^{(k+1)}] + \frac{1}{2}E[\sigma^{(k)}] + \frac{1}{2}E[\sigma^{(k+1)}]) \tag{2.29}$$

This allows us to write the partition function as:

$$Z(\beta) = \sum_{\sigma^{(1)}, \cdots, \sigma^{(m)}} \langle \sigma^{(1)} | \, T \, | \sigma^{(2)} \rangle \langle \sigma^{(2)} | \, T \, | \sigma^{(3)} \rangle \cdots \langle \sigma^{(m)} | \, T \, | \sigma^{(1)} \rangle = \mathrm{Tr}\, T^m \tag{2.30}$$

Observe that T is by definition symmetric, so it is necessarily diagonalizable with a collection of 2^n (not necessarily distinct) eigenvalues $\lambda_0 \geq \cdots \geq \lambda_{2^n-1}$. Thus, we can express the

partition function entirely in terms of a sum of eigenvalues:

$$Z(\beta) = \sum_{i=0}^{2^n-1} \lambda_i^m \tag{2.31}$$

Claim: A Simple Expression for the Free Energy

The free energy per site is given in the thermodynamic limit $(m, n \to \infty)$ as $f(\beta) = -\frac{k}{\beta} \lim_{n\to\infty} \frac{\log \lambda_0}{n}$, where λ_0 is the largest eigenvalue of T.

Proof. From the definition of free energy, it follows that:

$$f(\beta) = -\lim_{m,n\to\infty} \frac{k}{\beta}\frac{1}{mn} \log Z(\beta) = -\lim_{m,n\to\infty} \frac{k}{\beta}\frac{1}{mn} \log \left(\sum_{i=1}^{2^n-1} \lambda_i^m \right)$$

Now, by the **Perron-Frobenius Theorem** [17], since T has all strictly positive entries (as they are all of the form $\exp(\cdots)$), it must have a **unique** maximal eigenvalue λ_0. Then, we can write the free energy as:

$$-\lim_{m,n\to\infty} \frac{k}{\beta}\frac{1}{mn} \log \left(\lambda_0^m \left(1 + \sum_{i=1}^{2^n-1} \left(\frac{\lambda_i}{\lambda_0}\right)^m \right) \right)$$

Then, since λ_0 is strictly greater than every other eigenvalue of T, we have that: $\lim_{m\to\infty} \left(\frac{\lambda_i}{\lambda_0}\right)^m = 0$ for all $i > 1$. Thus, our expression simplifies as:

$$-\lim_{m,n\to\infty} \frac{k}{\beta}\frac{1}{mn} \log (\lambda_0^m)$$

From here, it immediately follows that:

$$f(\beta) = -\frac{k}{\beta} \lim_{n\to\infty} \frac{\log \lambda_0}{n}$$

as desired. □

As we can see, by employing the transfer matrix, much of the computations of observable quantities in this model have been reduced to a matter of finding the eigenvalues of an arbitrarily large matrix.

The transfer matrix approach's value far exceeds that of a cheeky computational tool, however. It is in fact a mathematically rigorous, discrete version of the infamous path integral from quantum field theory; we will see later in this chapter, in fact, that a certain limit of the transfer matrix exactly reproduces the time evolution of a certain quantum mechanical system. We interpret taking the trace over all row configurations of the transfer matrix as a weighted sum over all possible ways that the transfer matrix evolves one row into the next.

We can actually use this interpretation of the transfer matrix as the evolution of the system in the vertical direction to compute important physical quantities. Making use of the operator

$\hat{\sigma}_i$ which acts on row configurations and measures the spin of the i'th site via $\hat{\sigma}_i \left|\sigma_1 \cdots \sigma_n\right\rangle = \sigma_i \left|\sigma_1 \cdots \sigma_n\right\rangle$, we calculate the magnetization M as:

$$\begin{aligned}\langle\sigma_{11}\rangle &= \lim_{m\to\infty} \left(\operatorname{Tr} T^m\right)^{-1} \operatorname{Tr}\left(\hat{\sigma}\, T^m\right) \\ &= \lim_{m\to\infty} \sum_l \left(\frac{\lambda_l}{\lambda_0}\right)^m \langle 0|\hat{\sigma}_1|l\rangle \\ &= \langle 0|\hat{\sigma}_1|0\rangle \end{aligned} \tag{2.32}$$

where $|l\rangle$ refers to a member of the orthonormal basis of eigenvectors of the transfer matrix with eigenvalue λ_l, and λ_0 is the unique largest eigenvalue of the transfer matrix T, which dominates as m becomes large.

We can also consider the connected correlation function of two spins $\sigma_{1,1}$ and $\sigma_{1,\,1+r}$ separated by a vertical distance r in the thermodynamic limit. The pair correlation function is given by:

$$\begin{aligned}\langle\sigma_{11}\sigma_{1+r,1}\rangle &= \lim_{m\to\infty} \left(\operatorname{Tr} T^m\right)^{-1} \operatorname{Tr}\left(T^{m-r}\hat{\sigma}_1 T^r \hat{\sigma}_1\right) \\ &= \lim_{m\to\infty} \frac{1}{\lambda_0^m} \sum_l \langle 0|\lambda_0^{m-r}\hat{\sigma}_1|l\rangle\langle l|\lambda_l^r\hat{\sigma}_1|0\rangle \\ &= \langle\sigma_{11}\rangle^2 + |\langle 0|\hat{\sigma}_1|1\rangle|^2 \left(\frac{\lambda_1}{\lambda_0}\right)^r + \cdots \end{aligned} \tag{2.33}$$

where the remaining terms fall off as r increases. From this, along with our expression for the magnetization in Eq. 2.32, we find that the connected correlation function for the two spins separated by a vertical distance r is:

$$\langle\sigma_{11}\sigma_{1+r,1}\rangle_c = |\langle 0|\hat{\sigma}_1|1\rangle|^2 \left(\frac{\lambda_1}{\lambda_0}\right)^r + \cdots \tag{2.34}$$

Thus, for large r, the connected correlation function decays as

$$\langle s_{11}s_{1+r,1}\rangle_c \sim \left(\frac{\lambda_1}{\lambda_0}\right)^r = e^{-r/\xi}, \quad \text{with} \quad \xi := \left(\log\frac{\lambda_0}{\lambda_1}\right)^{-1}. \tag{2.35}$$

The *correlation length* ξ is the natural length scale of the Ising model; it roughly tells us how far changes in spins propgate throughout the system.

2.5 Crash Course in Quantum Statistical Mechanics

Before we delve into a few of the many reasons that over 800 papers on the Ising model are published each year, we must briefly overview the foundations of quantum (statistical) mechanics.

In the preceding sections, each microstate of our system was characterized by a single, definite value. However, at smaller length scales, quantum effects take over and there is an unavoidable **indeterminacy** introduced to our system in addition to the usual statistical uncertainty. By this, I mean the following:

1. States of the system no longer have a single value but are instead in a superposition of possible states with a probability of being in any given one of them.
2. The system evolves over time in ways which conserve the total probability.
3. When measured, the state collapses from a superposition onto a definite state.

The most natural mathematical arena in which all of these physical observations can be rigorously defined is a **Hilbert Space**.

Definition 2.5.1: Hilbert Space

A (finite dimensional) Hilbert Space is a vector space $\mathcal{H}$ over $\mathbb{C}$ with an inner product $\langle\cdot|\cdot\rangle : H \times H \longrightarrow H$ that obeys the following properties:

1. $\langle v|w\rangle = \overline{\langle w|v\rangle}$
2. $\langle v|aw_1 + bw_2\rangle = a\langle v|w_1\rangle + b\langle v|w_2\rangle \ \forall\ a, b \in \mathbb{C}$
3. $\langle v|v\rangle \geq 0$, with the equality only holding when $v = 0$.

Similarly to the classical case, we then want to define a Hamiltonian on our Hilbert Space of states. We define a **Hamiltonian** $H : \mathcal{H} \longrightarrow \mathcal{H}$ to be a self-adjoint (Hermitian) operator whose eigenvalues correspond to the energy of a given state. The motivation for this is as follows: by Spectral Theorem, since H is self-adjoint, it must have an orthonormal basis (ONB) of eigenvectors, $\{|\psi_i\rangle\}_i$, such that

$$H\,|\psi_i\rangle = E_i\,|\psi_i\rangle \quad \text{with} \quad E_i \in \mathbb{R} \tag{2.36}$$

We refer to these as the energy states of the system. Then, any vector $|\Psi\rangle \in \mathcal{H}$ can be expressed in terms of this ONB of energy states as $|\Psi\rangle = \sum_i a_i\,|\psi_i\rangle$ for some $a_i \in \mathbb{C}$. Naturally, then, we have:

$$\begin{aligned}
\langle\Psi|\,H\,|\Psi\rangle &= \sum_{i,\,j} \overline{a_i} a_j\,\langle\psi_i|\,H\,|\psi_j\rangle \\
&= \sum_{i,\,j} \overline{a_i} a_j\,\langle\psi_i|\,E_j\,|\psi_j\rangle \\
&= \sum_{i,\,j} \overline{a_i} a_j E_j \delta_{ij} \\
&= \sum_i |a_i|^2 E_i
\end{aligned} \tag{2.37}$$

Assuming $|\Psi\rangle$ is a unit vector, we have $\sum_i |a_i|^2 = 1$, giving $|a_i|^2$ the interpretation of the probability of the quantum state $|\Psi\rangle$ being measured in the state $|\psi_i\rangle$ with energy E_i. This means that $\langle\Psi|\,H\,|\Psi\rangle$ can be interpreted as the expected value of the energy of the quantum state $|\Psi\rangle$.

Now, we expect states to evolve in such a way that the probabilities still sum to 1 at any point in time, i.e the norm of the state vector remains unchanged. This implies that quantum

states should evolve in time by being acted on by a unitary operator $U(t)$ such that:

$$U(t)\,|\Psi(t_0)\rangle = |\Psi(t+t_0)\rangle \tag{2.38}$$

We should also have that $U(t_1)U(t_2) = U(t_1+t_2)$ for all $t \in \mathbb{R}$. Physically, we know that the energy of the state should be conserved through time, meaning it should not matter whether we measure the energy and then evolve the state or swap the order. Mathematically, this should imply:

$$[H, U(t)] := H\,U(t) - U(t)\,H = 0 \tag{2.39}$$

for all $t \in \mathbb{R}$. With these properties in mind, the most natural expression for the time evolution operator is:

$$U(t) = e^{iHt} \tag{2.40}$$

where the i is in the exponent to enforce unitarity, since we know that H is required to be Hermitan by definition.

From here, we have all of the necessary ingredients to do quantum statistical mechanics. We define the partition function of a quantum statistical mechanical system in analogy with the classical case by summing over the energy eigenvalues:

$$Z(\beta) = \sum_i e^{-\beta E_i} = \sum_i \langle\psi_i|\, e^{-\beta H}\, |\psi_i\rangle = \operatorname{Tr}\rho \tag{2.41}$$

where $\rho := e^{-\beta H}$ is referred to as the **density operator** and β is the inverse temperature. Take note of the similarities between ρ and the time evolution operator $U(t)$; specifically, they are related by the transformation $t \mapsto i\beta$, which is known as a *Wick rotation*. Thus, we think of the density operator ρ as giving the *imaginary time* (or Euclidean time) evolution of a quantum statistical mechanical system as the temperature is varied.

2.6 From Classical to Quantum

Given the similarity between the expression of the partition function in terms of transfer matrix (Eq. 2.30) for the 2d Ising model and the partition function for a quantum system (Eq. 2.41), one might wonder whether we can find a quantum mechanical system whose partition function is equivalent to the 2d classical Ising model's. This is indeed true, but there is some nuance required to reach this equivalence.

Consider a single row $\sigma^{(k)}$ of the 2d classical Ising model and its corresponding Hilbert space of configurations V. Observe that we can express V_i as being built out of n 2-dimensional Hilbert spaces – one for each site in the row. The Hilbert space of the site in the i'th column will be referred to as $\mathbb{C}_i^2$ and will be expressed in terms of the orthonormal basis vectors $|-\rangle$ and $|+\rangle$, such that the total Hilbert space is:

$$V = \bigotimes_i \mathbb{C}_i^2 \tag{2.42}$$

with the resulting basis vectors of the total space of row configurations being given by:

$$\{|+\rangle|+\rangle\cdots|+\rangle\,,\ |-\rangle|+\rangle\cdots|+\rangle\,,\ \cdots\,,\ |-\rangle|-\rangle\cdots|-\rangle\} \tag{2.43}$$

We choose the ONB $\{|-\rangle\,,\ |+\rangle\}$ on each site to be the eigenstates of the Pauli-Z matrix, where we recall the Pauli Matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.44}$$

We want to express the transfer matrix T (Eq. 2.29) in a new way which makes its relation with the quantum mechanical formulation of the partition function clear. First, we will split the transfer matrix into two pieces – T_k, which depends only on the energy within a single row, and $T_{k+\frac{1}{2}}$, which depends on the interaction between two neighboring rows.

Consider the matrix T_k which acts on the Hilbert space of the k'th row by acting on a given configuration with an eigenvalue corresponding to its energy. Denoting $\sigma_i^z = 1 \otimes \cdots \otimes \sigma^z \otimes \cdots \otimes 1$ with the σ^z in the i'th position, we can cleverly write T_k as:

$$T_k = \exp\left(K \sum_{i=1}^{n} \sigma_i^z \sigma_{i+1}^z\right) \tag{2.45}$$

Then, we can define another matrix $T_{k+\frac{1}{2}}$ which is applied after T_k and takes in a row configuration σ^k on the right and a configuration of the neighboring row σ^{k+1} on the left to spit out the interaction energy between them as matrix elements:

$$T_{k+\frac{1}{2}} = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \tag{2.46}$$

Figure 2.6: Visualization of the Transfer Matrix Between Rows k and $k+1$

It would be nice now if we could write the above expression for $T_{k+\frac{1}{2}}$ as the exponential of a sum of local Hermitian operators, like we did for T_k in Eq. 2.46. Observe that:

$$\begin{aligned} e^{K'\sigma^x} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & K' \\ K' & 0 \end{pmatrix} + \begin{pmatrix} \frac{K'^2}{2} & 0 \\ 0 & \frac{K'^2}{2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{K'^3}{6} \\ \frac{K'^3}{6} & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + \frac{K'^2}{2} + \cdots & K' + \frac{K'^3}{6} + \cdots \\ K' + \frac{K'^3}{6} + \cdots & 1 + \frac{K'^2}{2} + \cdots \end{pmatrix} \\ &= \begin{pmatrix} \cosh K' & \sinh K' \\ \sinh K' & \cosh K' \end{pmatrix} \end{aligned} \tag{2.47}$$

Noticing that Eq. 2.47 looks similar to one of the tensor factors in Eq. 2.46, we may wonder

whether there exists K' such that:

$$\begin{pmatrix} \cosh K' & \sinh K' \\ \sinh K' & \cosh K' \end{pmatrix} \propto \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix} \tag{2.48}$$

Dividing the two independent constraints, we see that such a K' would satisfy: $\tanh K' = e^{-2K}$, or $-\frac{1}{2}\log\tanh K' = K$. This should remind you of the expression of the dual coupling from Kramers Wannier duality (Eq. 2.22), as we in fact have that $K = \tilde{K}'$! This makes sense considering our interpretation in Figure 2.6 of $T_{k+\frac{1}{2}}$ acting on the dual lattice between two interacting rows. Ultimately, after some algebra, we find the proportionality constant and obtain:

$$\begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix} = \sqrt{2\sinh 2\tilde{K}'}\exp\left(K'\sigma^{x}\right) \tag{2.49}$$

Plugging this in for each factor in Eq. 2.46, we obtain:

$$T_{k+\frac{1}{2}} = (2\sinh 2\tilde{K}')^{\frac{n}{2}}\exp\left(K'\sum_{i=1}^{n}\sigma_i^{x}\right) \tag{2.50}$$

where $\sigma_i^x = 1\otimes\cdots\otimes\sigma^x\otimes\cdots\otimes 1$ with σ^x in the i'th position. From here, we finally can express the transfer matrix as[1]:

$$T = T_{k+\frac{1}{2}}T_k = (2\sinh 2\tilde{K}')^{\frac{n}{2}}\exp\left(K'\sum_{i=1}^{n}\sigma_i^{x}\right)\exp\left(\tilde{K}'\sum_{i=1}^{n}\sigma_i^{z}\sigma_{i+1}^{z}\right) \tag{2.51}$$

Then, by the same logic as we used a couple of sections ago, the partition function will be given by the trace of T^m as in Eq. 2.30. However, since the partition function is only defined up to an overall constant, we can safely drop the factor of $(2\sinh 2\tilde{K}')^{\frac{n}{2}}$ from each transfer matrix to obtain a nicer expression for Z:

$$Z(K') = \mathrm{Tr}\left(\exp\left(K'\sum_{i=1}^{n}\sigma_i^{x}\right)\exp\left(\tilde{K}'\sum_{i=1}^{n}\sigma_i^{z}\sigma_{i+1}^{z}\right)\right)^{m} \tag{2.52}$$

From here, we would like to use the additive property of the exponential function and write the partition function as the trace of the exponential of a single Hamiltonian, but unfortunately because σ^x and σ^z do not commute, this is not so simple. Indeed, writing $A := \sum_{i=1}^{n}\sigma_i^x$ and $B := \sum_{i=1}^{n}\sigma_i^z\sigma_{i+1}^z$, we can expand out the exponentials to see where our procedure breaks down:

$$\exp\left(K'A\right)\exp\left(\tilde{K}'B\right) = \left(1 + K'A + \cdots\right)\left(1 + \tilde{K}'B + \cdots\right)$$

$$= \left(1 + K'A + \tilde{K}'B + \frac{K'\tilde{K}'}{2}(AB + BA) + \cdots\right) + \left(\frac{K'\tilde{K}'}{2}[A,B] + \cdots\right) \tag{2.53}$$

Thus, our partition function can be expressed as:

$$Z(K') = \left(\exp\left(K'A + \tilde{K}'B\right) + \mathcal{O}(K'\tilde{K}'\,[A,B])\right)^{m} \tag{2.54}$$

[1]Note that this is not exactly equal to Eq. 2.29, but it will ultimately lead to an equivalent partition function.

It is reasonable to assume that if K', $\tilde{K}' \ll 1$ in some limit, then the terms with the commutator terms will vanish. However, by Kramers-Wannier duality, if we try to send K' to 0, then $\tilde{K}'$ will be sent to ∞, which makes this even trickier. Let's speculate for a moment about why we have reached this impasse.

Recall in the last section that that the density operator $\rho = e^{-\beta H}$ is in fact just a Wick rotated time evolution operator. Meanwhile, the transfer matrix T – which we are trying to equate with a density operator – can be thought of as providing the evolution of the Ising model from one row to the next. Thus, the issue likely lies in the fact that we are trying to equate a discrete evolution operator, T, to a continuous time evolution operator, ρ. We may expect that if we can find a way of implementing a suitable limit that makes the spacing between rows go to 0, then we can equate the vertical axis of the 2d classical model with the Euclidean time in a 1d quantum model and reach some kind of equivalence between T and ρ.

Since we are going to compress the lattice in the vertical direction so that we no longer have a square lattice, let us assume that the coupling between rows and columns is no longer equal. For instance, let's make the replacement $K \mapsto K_x$ for the coupling in T_k (Eq. 2.46) and replace $K \mapsto K_t$ in $T_{k+\frac{1}{2}}$ (Eq. 2.47). With these replacements, our transfer matrix becomes:

$$T = \exp\left(K_t' \sum_{i=1}^{n} \sigma_i^x\right) \exp\left(K_x \sum_{i=1}^{n} \sigma_i^z \sigma_{i+1}^z\right) \tag{2.55}$$

where K_t' satisfies $\tanh K_t' = e^{-2K_t}$ as before. Physically, we expect that as the rows get closer together we should have $K_t \gg K_x$, i.e the vertical coupling becomes stronger by comparison. Thus, if we take the limit as the spacing between rows becomes infinitesimally small, we should have $K_t \to \infty$ as the number of rows, m, also increases to ∞ in a corresponding way.

From the expression above, observe that $K_t' \to 0$ as e^{-2K_t}. To maintain some sense of isotropy, we also want $K_x \to 0$ correspondingly, so we assume that

$$K_x = \lambda e^{-2K_t} \tag{2.56}$$

where $\lambda \in \mathbb{R}$ is a coupling constant. We interpret e^{-2K_t} as the lattice spacing, which shrinks as we send $K_t \to \infty$. When we take the continuum limit, we want to ensure that as we increase the number of rows m to infinity and shrink the lattice spacing to 0, the overall vertical size of the system remains constant. In other words, we should have:

$$m \cdot e^{-2K_t} = \overline{\beta} \tag{2.57}$$

where $\overline{\beta}$ is a fixed constant. Following this logic, we have:

$$\lim_{K_t \to \infty} T = \exp\left(\frac{\overline{\beta}}{m} \sum_{i=1}^{n} \sigma_i^x\right) \exp\left(\frac{\overline{\beta}\lambda}{m} \sum_{i=1}^{n} \sigma_i^z \sigma_{i+1}^z\right) \tag{2.58}$$

Then, using the Trotter Identity,

$$\lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n = e^{A+B} \tag{2.59}$$

we have that:

$$\begin{aligned} Z &= \lim_{m\to\infty} \mathrm{Tr}\left(\exp\left(\frac{\overline{\beta}}{m}\sum_{i=1}^{n}\sigma_i^x\right)\exp\left(\frac{\overline{\beta}\lambda}{m}\sum_{i=1}^{n}\sigma_i^z\sigma_{i+1}^z\right)\right)^m \\ &= \mathrm{Tr}\,\exp\overline{\beta}\left(\sum_{i=1}^{n}\sigma_i^x + \lambda\sum_{i=1}^{n}\sigma_i^z\sigma_{i+1}^z\right) \end{aligned} \tag{2.60}$$

Thus, in the Euclidean time-continuum limit, we have an equivalence between the partition function of the classical 2d Ising model in the transfer matrix formalism and the partition function of a 1d quantum mechanical model. Specifically, we have

$$Z = \lim_{m\to\infty}\mathrm{Tr}\,T^m = \mathrm{Tr}\,\rho \tag{2.61}$$

where $\rho = e^{-\overline{\beta}H}$ is the density operator with (quantum) inverse temperature $\overline{\beta}$ corresponding to the Hamiltonian:

$$H = -\sum_{i=1}^{n}\sigma_i^x - \lambda\sum_{i=1}^{n}\sigma_i^z\sigma_{i+1}^z \tag{2.62}$$

Traditionally, the coupling λ is absorbed into $\overline{\beta}$ and a new coupling $h := \frac{1}{\lambda}$ is introduced so that we have:

$$H = -\sum_{i=1}^{n}\sigma_i^z\sigma_{i+1}^z - h\sum_{i=1}^{n}\sigma_i^x \tag{2.63}$$

as our Hamiltonian. Either way, we refer to the corresponding 1d quantum mechanical system as the **1d transverse field Ising model** (1d TFI) as the coupling term h plays the role of a variable magnetic field that we saw in previous sections.

It's important to note that this procedure is not specific to the 2d Ising model. In fact, a $(d+1)$ dimensional classical statistical mechanical model will be equivalent to a corresponding d dimensional quantum mechanical model in the Euclidean time-continuum limit, so long as the partition function obeys a property known as **Reflection Positivity** [18].

Chapter 3

The 1d Transverse Field Ising Model

In the last chapter, we found that we can express the partition function of the 2d classical Ising model as the trace over repeated applications of the transfer matrix. We further posited that the transfer matrix, which acts on the Hilbert space of row configurations

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathbb{C}_i^2 \tag{3.1}$$

has a natural interpretation as an evolution operator which takes one row configuration (a basis vector in $\mathcal{H}$) to the one above it. Thinking of the vertical axis of the 2d Ising model as a discrete notion of time, we drew an analogy between the classical transfer matrix and the quantum time evolution operator in quantum mechanics. Finally, we were able to solidify these analogies by taking a suitable anisotropic limit as the spacing between adjacent rows goes to 0 – the so-called Euclidean time continuum limit.

Specifically, we found the equivalence:

$$Z = \operatorname{Tr} T^m \xrightarrow[m\to\infty]{\sim} \operatorname{Tr} e^{-\overline{\beta}H} \tag{3.2}$$

where H is the Hamiltonian

$$H = -\sum_{i=1}^{n} \sigma_i^z \sigma_{i+1}^z - \lambda \sum_{i=1}^{n} \sigma_i^x \tag{3.3}$$

which acts on a 1d chain of spins, whose possible quantum states are given by the Hilbert space $\mathcal{H}$. This classical-quantum correspondence is incredibly powerful, as the effective inverse temperature $\overline{\beta}$ is proportional to the vertical size of the 2d Ising model (recall Eq. 2.57). This means that taking the thermodynamic limit as the size of the classical model becomes infinite involves sending $\overline{\beta} \to \infty$ (i.e sending the effective temperature of the quantum system to 0). Using this, we can actually find an exact closed-form expression for the partition function in the thermodynamic limit. Indeed, if E_i are the eigenvalues of the

Hamiltonian and g_i are the degeneracies of these eigenvalues, we have:

$$\begin{aligned}\lim_{\overline{\beta}\to\infty} \operatorname{Tr} e^{-\overline{\beta}H} &= \lim_{\overline{\beta}\to\infty} \sum_{i=0}^{\infty} g_i e^{-\overline{\beta}E_i} \\ &= \lim_{\overline{\beta}\to\infty} e^{-\overline{\beta}E_0}\left(g_0 + \sum_i g_i e^{\overline{\beta}(E_0-E_i)}\right) \\ &= g_0 \lim_{\overline{\beta}\to\infty} e^{-\overline{\beta}E_0} \end{aligned} \tag{3.4}$$

where E_0 is the smallest eigenvalue of H, which we call the **ground state energy**.

This spectral interpretation also reveals the origin of the correlation length in the classical model. Let us consider the connected correlation function of two spins separated by a vertical distance r. From our previous discussion of the correlation function along with our knowledge of the transfer matrix's relation to the time evolution operator, we find:

$$\langle \sigma_i \sigma_{i+r} \rangle_c = \sum_{n\geq 1} |\langle 0|\sigma_i^z|n\rangle|^2 e^{-r(E_n - E_0)}. \tag{3.5}$$

For large separations r, the sum is dominated by the first excited state. Therefore, the connected correlation decays exponentially:

$$\langle \sigma_i \sigma_{i+r} \rangle_c \sim e^{-r/\xi}, \quad \text{with} \quad \xi := \frac{1}{E_1 - E_0}. \tag{3.6}$$

From this observation, we can see that understanding the ground states of the 1d quantum Hamiltonian is crucial to understanding the behavior of the 2d Ising model in the thermodynamic limit.

3.1 Symmetries and their Breaking

Given that we obtained the 1d transverse-field Ising model directly from an extreme anisotropic limit of the 2d classical Ising model, we may expect that the symmetries of the original model – the global $\mathbb{Z}_2$ symmetry and Kramers-Wannier duality – manifest somehow in the quantum case. This is indeed true, and these symmetries are in fact easier to see in this model. As we constructed our Hamiltonian to be diagonalized in terms of the eigenstates of σ^z, the global spin-flip operator will be given by:

$$S = \prod_{i=1}^{n} \sigma_i^x \tag{3.7}$$

Observe that S commutes with our Hamiltonian, as:

$$[S, H] = -\sum_{j=1}^{n} [\prod_{i=1}^{n} \sigma_i^x, \sigma_j^z \sigma_{j+1}^z] - h \sum_{j=1}^{n} [\prod_{i=1}^{n} \sigma_i^x, \sigma_j^x] \to 0$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\prod_{i=1}^{n}\sigma_i^x\sigma_j^z\sigma_{j+1}^z + \sum_{j=1}^{n}\sigma_j^z\sigma_{j+1}^z\prod_{i=1}^{n}\sigma_i^x \\
&= -\sum_{j=1}^{n}(-\sigma_j^z\prod_{i=1}^{n}\sigma_i^x)\sigma_{j+1}^z + +\sum_{j=1}^{n}\sigma_j^z\sigma_{j+1}^z\prod_{i=1}^{n}\sigma_i^x \\
&= \sum_{j=1}^{n}\sigma_j^z(-\sigma_{j+1}^z\prod_{i=1}^{n}\sigma_i^x) + \sum_{j=1}^{n}\sigma_j^z\sigma_{j+1}^z\prod_{i=1}^{n}\sigma_i^x \\
&= 0
\end{aligned} \tag{3.8}
$$

where we used the fact that $\sigma_i^x\sigma_j^z = -\sigma_j^z\sigma_i^x$ if $i = j$ and $\sigma_i^x\sigma_j^z = \sigma_j^z\sigma_i^x$ otherwise. We interpret Eq.**??** as saying that the energy of a system is the same before or after a global spin flip is applied; hence, we say that S is a symmetry of the Hamiltonian.

Because S commutes with the Hamiltonian, they can be simultaneously diagonalized; i.e there exists a basis of energy eigenstates that are also eigenstates of the symmetry operator. However, this does not mean that any eigenstate of H will also be an eigenstate of S, as our Hamiltonian can have degenerate eigenvalues. To see this, first consider our Hamiltonian in the limit of very large magnetic field ($h \to \infty$), in which case the σ^x terms dominate and we have:

$$H \approx -h\sum_{i=1}^{n}\sigma_i^x \tag{3.9}$$

In this limit, we have a single unique ground state:

$$|0\rangle = \frac{1}{2^{\frac{n}{2}}}(|+\rangle + |-\rangle) \otimes \cdots \otimes (|+\rangle + |-\rangle) \tag{3.10}$$

with ground state energy given by:

$$H|0\rangle = -nh|0\rangle \tag{3.11}$$

It is easy to see that in this limit, the ground state $|0\rangle$ is indeed symmmetric under the global $\mathbb{Z}_2$ symmetry, as:

$$S|0\rangle = \frac{1}{2^{\frac{n}{2}}}\sigma^x(|+\rangle + |-\rangle) \otimes \cdots \otimes \sigma^x(|+\rangle + |-\rangle) = |0\rangle \tag{3.12}$$

Now, consider the limit of the Hamiltonian in which there is no magnetic field ($h = 0$); in this case, we have:

$$H = -\sum_{i=1}^{n}\sigma_i^z\sigma_{i+1}^z \tag{3.13}$$

Here, we see that in order for each term $-\sigma_i^z\sigma_{i+1}^z$ to contribute a -1 eigenvalue, each pair of neighboring spins must be aligned. This means that we have **two unique ground states**:

$$|0\rangle_\uparrow = |+\rangle \otimes \cdots \otimes |+\rangle \qquad |0\rangle_\downarrow = |-\rangle \otimes \cdots \otimes |-\rangle \tag{3.14}$$

While the equal superposition of the two ground states is indeed symmetric under the $\mathbb{Z}_2$

action,

$$S\left(\frac{|0\rangle_\uparrow + |0\rangle_\downarrow}{\sqrt{2}}\right) = \frac{|0\rangle_\uparrow + |0\rangle_\downarrow}{\sqrt{2}} \tag{3.15}$$

upon measuring the ground state to be in either the up or down state, we find that the symmetry is broken, as

$$S\,|0\rangle_\uparrow = |0\rangle_\downarrow \qquad S\,|0\rangle_\downarrow = |0\rangle_\uparrow \tag{3.16}$$

This phenomenon in which choosing a specific ground state breaks a symmetry of its Hamiltonian is referred to as **spontaneous symmetry breaking**. In fact, it is through this mechanism that the existence of the long-elusive Higgs boson was predicted!

Now, let us inspect the Hamiltonian for any possible remnants of Kramers-Wannier duality that are leftover from the classical case. Motivated by our finding from our original derivations, we will try to phrase our Hamiltonian in terms of variables defined on the links between the sites of our original 1d lattice. First, notice that each of the operators $\sigma_i^z\sigma_{i+1}^z$ is specified by a single link connecting the sites i and $i+1$; we will quantify this by defining the operator:

$$\mu^x_{i+\frac{1}{2}} = \sigma_i^z\sigma_{i+1}^z \tag{3.17}$$

which is simply the quantum analog of the k_l variables that we defined on the links of the classical model. $\mu^x_{i+\frac{1}{2}}$ has eigenvalues of $+1$ and -1 depending on whether the spins at the two ends of the link are aligned or not. With this in mind, we once again define a dual lattice with $n-1$ sites which can be seen below in Figure 3.1. Observe that acting with σ_i^x

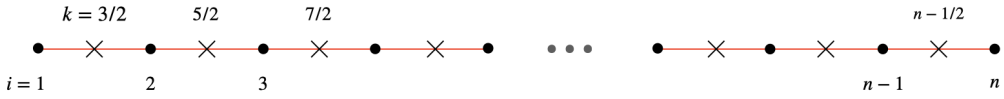


Figure 3.1: Lattice and Dual Lattice for 1d Quantum Ising Model

on our original lattice will disrupt both the measurements of $\mu^x_{i-\frac{1}{2}}$ and $\mu^x_{i+\frac{1}{2}}$, as:

$$\{\sigma_i^x, \mu^x_{i\pm\frac{1}{2}}\} = \{\sigma_i^x, \sigma_i^z\sigma_{i\pm1}^z\} = 0 \tag{3.18}$$

since $\{\sigma_i^x, \sigma_i^z\} = 0$ and $[\sigma_i^x, \sigma_{i\pm1}^z] = 0$ (where $\{A, B\} = AB + BA$ is the anticommutator). Using this property, we'd like to construct an operator which creates an excitation on a single link – mapping $\mu^x_{i+\frac{1}{2}} \mapsto -\mu^x_{i+\frac{1}{2}}$ for a given i while leaving the rest fixed. Since the application of a single σ_i^x disrupts the link on $i+\frac{1}{2}$ and $i-\frac{1}{2}$, we can imagine trying to undo one of these disruptions by applying σ^x_{i-1} but this will also flip $i-\frac{3}{2}$ in the process, unfortunately. However, we can imagine continuing this process – not forever, but just until we hit the boundary. Since site $i=1$ is only connected to a single link, σ_i^x will just revert the leftmost link to finally leave only the excitation on $i+\frac{1}{2}$. We formalize this by defining the operator:

$$\mu^z_{i+\frac{1}{2}} = \prod_{j=1}^{i}\sigma_j^x \tag{3.19}$$

which acts with σ_j^x on every site to the left of the link $i+\frac{1}{2}$. The action of $\mu^z_{i+\frac{1}{2}}$ is visualized below in Figure 3.2.

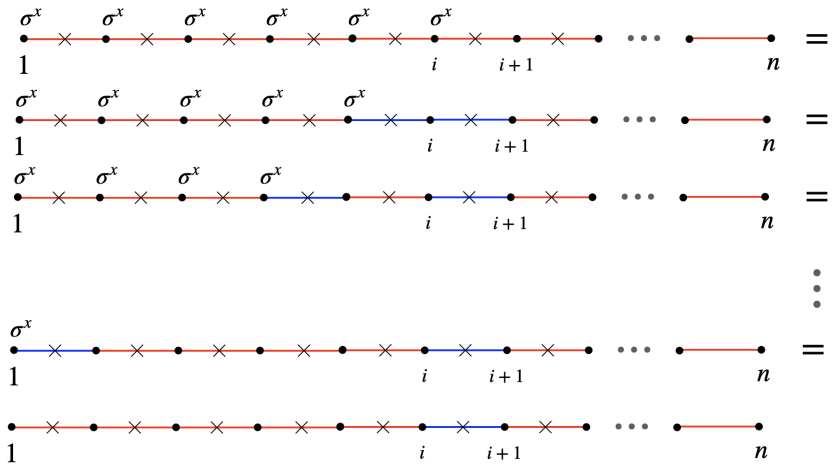


Figure 3.2: Action of $\mu^z_{i+\frac{1}{2}}$ on the dual lattice

This operator is often referred to as a **kink creation operator**, as it is an extended operator that creates an excitation on just a single link. Notice that our new operators obey

$$\{\mu^z_{i+\frac{1}{2}}, \mu^x_{i+\frac{1}{2}}\} = 0 \tag{3.20}$$

while if $i \neq j$, we have:

$$[\mu^z_{i+\frac{1}{2}}, \mu^x_{j+\frac{1}{2}}] = 0 \tag{3.21}$$

Also, observe that:

$$\begin{aligned} \mu^z_{i-\frac{1}{2}}\mu^z_{i+\frac{1}{2}} &= \prod_{j<i}\prod_{k<i+1} \sigma^x_j \sigma^x_k \\ &= \sigma^x_i \prod_{j<i} (\sigma^x_j)^2 \\ &= \sigma^z_i \end{aligned} \tag{3.22}$$

where we used the fact that $(\sigma^x)^2 = 1$. Now, using Eq. 3.17 and Eq. 3.22, we can rewrite our Hamiltonian in terms of our new dual variables:

$$\begin{aligned} H &= -h\sum_{i=2}^{n-1} \mu^z_{i-\frac{1}{2}}\mu^z_{i+\frac{1}{2}} - \sum_{i=2}^{n-1} \mu^x_{i-\frac{1}{2}} - h(\sigma^x_1 + \sigma^x_n) \\ &= h\left(-\sum_{i=2}^{n-1} \mu^z_{i-\frac{1}{2}}\mu^z_{i+\frac{1}{2}} - \frac{1}{h}\sum_{i=2}^{n-1} \mu^x_{i-\frac{1}{2}}\right) - h(\sigma^x_1 + \sigma^x_n) \end{aligned} \tag{3.23}$$

Observe that μ^z and μ^x not only have the same eigenvalues as σ^x and σ^z, but they also obey the exact same commutation relations as we discovered in Eq. 3.20 and Eq. 3.21! Thus, if we ignore the boundary terms on sites $i = 1$ and $i = n$ for now – and simply rename $\mu \mapsto \sigma$ – we can see that Eq. 3.23 is of the exact same form as the 1d TFI Hamiltonian that we started with! Precisely, we have that the bulk (the system excluding the boundary) energy spectrum obeys:

$$E(h) = hE(1/h) \tag{3.24}$$

Thus, we have a relation between the 1d quantum Ising model at weak coupling and strong coupling, which is indeed just Kramers-Wannier duality! However, there are a few subtleties here which we intentionally swept under the rug for the time being.

First, there is obviously the issue of boundary conditions. From Eq. 3.23, it is clear that different boundary conditions of the initial model will be mapped differently under the duality map. Even more damningly, how do we define the duality mapping if our model is defined on a ring, i.e if sites $i = 1$ and $i = n$ are linked? Our definition of μ_i^z relies on their being a boundary to reach, but how will this work on a closed surface?

Secondly, suppose that we want to look at the image of the Ising chain with $h \gg 1$ under the duality mapping. We know from our earlier discussion that in this limit, there are two ground states, $|0\rangle_\uparrow$ and $|0\rangle_\downarrow$. Under Kramers-Wannier duality, this model will be mapped to another Ising chain with $h \ll 1$, in which case we know there is a single unique ground state, $|0\rangle$. Consequently, the Kramers-Wannier transformation must map map two ground states into one, meaning it is not only not unitary, but it is not even invertible!

Although this may be uncomfortable for those accustomed to symmetries being described by groups, we will begin to explore how to cope with and appreciate the mathematical beauty of non-invertible symmetries starting in the next section. In fact, the rest of this thesis will be dedicated to examining the structure of non-invertible symmetries by focusing on the Kramers-Wannier transformation.

3.2 A First Look at Topological Defects

Following arguments presented in [19] and [7], we will discuss Kramers-Wannier duality in a modern context by introducing concepts like gauging and topological defects. We begin by putting our theory on a ring; specifically, we add an extra link on the right of site n and introduce the following boundary conditions:

$$|\sigma_{i+n}\rangle = (\sigma_i^x)^t |\sigma_i\rangle = |(-1)^t \sigma_i\rangle \tag{3.25}$$

where $|\sigma_i\rangle$ refers to the spin on the i'th site. When $t = 0$, we have periodic boundary conditions (PBC), and when $t = 1$, we have **twisted** boundary conditions (TBC). Taking into account these adjustments, our Hamiltonian becomes:

$$H = -\sum_{i=1}^{n} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{n} \sigma_i^x - (-1)^t \sigma_n^z \sigma_1^z \tag{3.26}$$

If $t = 1$, our Hamiltonian is modified from the usual Ising model and we say that there is a $\mathbb{Z}_2$ **defect** on the link $(n, 1)$, which we will call η. We will refer to the Hamiltonian with this defect as $H_\eta^{(n,1)}$. Now, observe what happens when we perform a basis change on this Hamiltonian by conjugating with σ_1^x:

$$\sigma_1^x H_\eta^{(n,1)} \sigma_1^x = \sigma_1^x (-\sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z + \sigma_n^z \sigma_1^z) \sigma_1^x - h \sum_{i=1}^{n} \sigma_i^x$$

$$= -\sum_{i=2}^{n-1} \sigma_i^z \sigma_{i+1}^z - \sigma_n^z \sigma_1^z - h \sum_{i=1}^{n} \sigma_i^x + \sigma_1^z \sigma_2^z = H_\eta^{(1,2)} \tag{3.27}$$

As we can see, the $\sigma_1^z \sigma_2^z$ term flipped its sign while the $\sigma_n^z \sigma_1^z$ term has reverted back to normal! Thus, by simply performing a basis change of our Hamiltonian – something that has absolutely zero impact on the energy spectrum and the overall physics of the theory – we have moved the $\mathbb{Z}_2$ defect from link $(n, 1)$ to link $(1, 2)$! It is clear that by repeatedly conjugating with various σ^x's we can continue to move this defect to any link on the lattice that we like without changing any of the underlying physics; because of this, we say that η is a **topological defect**. This twist or defect should be thought of as the non-trivial $\mathbb{Z}_2$ bundle over the circle (Fig. 3.3). The circle is twisted over itself, but the exact point at which the kink was initially does not change its shape in any way – we can move the kink to any point around the circle that we wish. Two defects on adjacent links can also interact

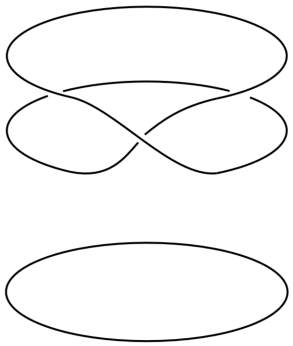


Figure 3.3: $\mathbb{Z}_2$ Principal Bundle over the Circle

with one another and fuse together. Indeed, consider the following Hamiltonian with defects on sites $(n, 1)$ and $(1, 2)$:

$$H_{\eta;\eta}^{(n,1);\,(1,2)} = -\sum_{i=2}^{n-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{n} \sigma_i^x + \sigma_n^z \sigma_1^z + \sigma_1^z \sigma_2^z \tag{3.28}$$

Now, observe that if we change basis by conjugating with σ_1^x on both sides, we obtain:

$$\sigma_1^x H_{\eta;\eta}^{(n,1);\,(1,2)} \sigma_1^x = -\sum_{i=2}^{n-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{n} \sigma_i^x - \sigma_n^z \sigma_1^z - \sigma_1^z \sigma_2^z = H \tag{3.29}$$

Thus, two neighboring $\mathbb{Z}_2$ defects can be annihilated by acting by conjugation with a σ^x on the site between them. We can also say that the two defects fuse together to produce the **trivial defect**. We will visualize both the movement operators and the fusion operators with the graphical calculus introduced below in Fig. 3.4.

Global symmetries are deeply related to topological defects; in fact, they are what makes defects topological. They constrain the large-scale behavior of a system by dividing the space of Hamiltonians into equivalence classes corresponding to the homotopy classes of topological defects. In fact, we can actually reconstruct a global symmetry of a system just by considering at a corresponding topological defect. For instance, consider starting from

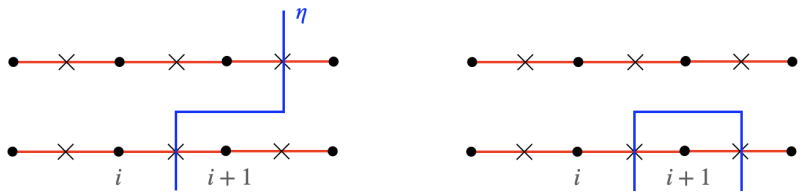


Figure 3.4: Action of conjugation by σ^x_{i+1} on $H_\eta^{(i,\,i+1)}$ (movement) and $H_{\eta;\,\eta}^{(i,\,i+1);\,(i+1,\,i+2)}$ (fusion)

the untwisted Hamiltonian and conjugating it with σ^x_1 to create defects on links $(n,1)$ and $(1,2)$:

$$\sigma^x_1 H \sigma^x_1 = -\sum_{i=2}^{n-1} \sigma^z_i \sigma^z_{i+1} - h\sum_{i=1}^{n} \sigma^x_i + \sigma^z_n \sigma^z_1 + \sigma^z_1 \sigma^z_2 \tag{3.30}$$

We can then move the defect on $(1,2)$ to $(2,3)$ by conjugating again by σ^x_2. Then, we can move this defect all the way to the link $(n-1,\ n)$by conjugating with σ^x_3, then by σ^x_4, and so on all the way up to σ^x_{n-1}. At this point, there are defects on $(n-1,\ n)$ and $(n,1)$, so conjugating once more by σ^x_n will annihilate these two defects and take us back to the original Hamiltonian. This can be summarized by the following equation:

$$(\sigma^x_n \cdots \sigma^x_1) H (\sigma_1 \cdots \sigma^x_n) = H \tag{3.31}$$

Recognizing the product of σ^x operators on both sides as the $\mathbb{Z}_2$ symmetry operator S from last section, this equation immediately implies:

$$[S, H] = 0 \tag{3.32}$$

which is exactly the statement that the spin flip operator S is a symmetry of the Hamiltonian! From this, we can see that global symmetries of the Hamiltonian and topological defects of the lattice model are really just two sides of the same coin; we can start from either and recover the other one.

3.3 Kramers-Wannier Duality from Gauging

Given the fact that topological defects live on the dual lattice and are intrinsically tied to the symmetries of a system, we may start to wonder if they can help us to better understand Kramers-Wannier duality in the 1d TFI.

Let us begin by extending our Hilbert space to include another spin that lives on each link of the lattice; we will refer to these as dual spins. Labeling the site on the link between i and $i+1$ as $i+\frac{1}{2}$, we define a Hilbert space on the links of the lattice as:

$$\tilde{\mathcal{H}} = \bigotimes_{j=1}^{n} \mathbb{C}^2_{j+\frac{1}{2}} \tag{3.33}$$

Then, the extended Hilbert space containing both the spins and dual spins will simply be

denoted by:

$$\overline{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}} \tag{3.34}$$

The spins on the links will likewise be considered in terms of the eigenbasis of $\tilde{\sigma}^z_{i+\frac{1}{2}}$ with $|+\rangle_{i+\frac{1}{2}}$ and $|-\rangle_{i+\frac{1}{2}}$ denoting the absence and presence of a defect on the link $i+\frac{1}{2}$, respectively. Now, we define a Hamiltonian on our extended Hilbert space which measures whether there is a defect on a given link and then acts with an appropriately modified Hamiltonian on the sites:

$$\overline{H} = \sum_{a_1,\cdots,a_n=\pm 1} H^{(1,2);\cdots;(n,1)}_{a_1;\cdots;a_n} \otimes |a_1 \cdots a_n\rangle \langle a_1 \cdots a_n| \tag{3.35}$$

This amounts to adding an additional minus sign in front of every interaction term in the Hamiltonian for which there is an η defect on the link between the two sites, which is exactly captured by the following expression:

$$\overline{H} = -\sum_{i=1}^{n} \sigma^z_i \tilde{\sigma}^z_{i+\frac{1}{2}} \sigma^z_{i+1} - h \sum_{i=1}^{n} \sigma^x_i \tag{3.36}$$

This process of summing over all possible defects on the links of our lattice is analogous to coupling a QFT to background $\mathbb{Z}_2$ gauge fields. Observe now that our new **gauged** Hamiltonian has a local symmetry for every site j of the original lattice generated by the operator $\mathcal{U}_j := \tilde{\sigma}^x_{j-\frac{1}{2}} \sigma^x_j \, \tilde{\sigma}^x_{j+\frac{1}{2}}$, as we have:

$$[\,\mathcal{U}_j\,, \overline{H}\,] = -\sum_{i=1}^{n} [\tilde{\sigma}^x_{j-\frac{1}{2}} \sigma^x_j \, \tilde{\sigma}^x_{j+\frac{1}{2}}, \sigma^z_i \tilde{\sigma}^z_{i+\frac{1}{2}} \sigma^z_{i+1}] = 0 \tag{3.37}$$

which follows from the fact that the only potentially non-commuting terms will share exactly two sites in common; thus, since $\{\sigma^z_i, \sigma^x_i\} = 0$, they will pick up two minus signs which cancel when they commute past each other.

The fact that we can choose to act with a $\mathcal{U}_j$ on every single site without impacting the energy of a configuration should less-so be thought of as a traditional symmetry which constrains our system, but rather as a **redundancy** of the system. We refer to this kind of symmetry as a **gauge symmetry** and the application of any number of $\mathcal{U}_j$'s as a **gauge transformation**, as their application does not affect the physics but simply changes the gauge in which we measure physical quantities.

But where does this redundancy come from? Recall that to introduce our extended, gauged, Hilbert space and its Hamiltonian we simply naively summed over all possible ways of putting topological defects on the links of our lattice. We did not take into account that many configurations are equivalent to others by either moving defects around or fusing defects together. Indeed, for each site of our original lattice, consider the following projection operator as defined by our graphical calculus:
We will refer to this projection as a Gauss law on site j, as it is reminiscent of Gauss's law from electromagnetism. This projection simply enforces that our Hilbert space is defined in such a way that incorporates the fusion of neighboring defects. Performing the sum over the delta function, we obtain:

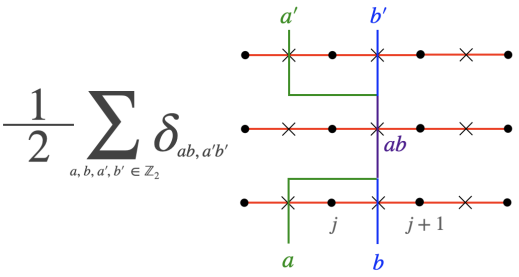


Figure 3.5: Projection operator for fusion redundancies on site j

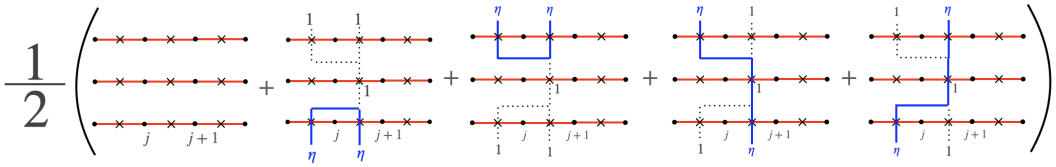


Figure 3.6: $\mathcal{G}_j$ simplified

In equation form, this is written as:

$$\begin{aligned}\mathcal{G}_j &= \frac{1}{2}\left(1 + \sigma_j^x \otimes \left(|{+}{+}\rangle\langle{-}{-}| + |{-}{-}\rangle\langle{+}{+}| + |{-}{+}\rangle\langle{+}{-}| + |{+}{-}\rangle\langle{-}{+}| \right)_{j-\frac{1}{2},\, j+\frac{1}{2}}\right) \\ &= \frac{1}{2}(1 + \tilde{\sigma}^x_{j-\frac{1}{2}} \sigma^x_j \, \tilde{\sigma}^x_{j+\frac{1}{2}}) \end{aligned} \tag{3.38}$$

Notice that the second term in the parentheses is exactly the gauge transformation $\mathcal{U}_j$ operator from earlier:

$$\mathcal{G}_j = \frac{1}{2}(1 + \mathcal{U}_j) \tag{3.39}$$

Thus, by enforcing the consistency of the fusion of consecutive defects by acting with the Gauss law projector, we are in fact just projecting our Hilbert space onto the subspace in which

$$\mathcal{U}_j \overline{\mathcal{H}} = \overline{H} \quad \text{for all} \quad j \tag{3.40}$$

In other words, by acting with the projector

$$\mathcal{G} = \prod_{j=1}^{n} \mathcal{G}_j \tag{3.41}$$

on our extended Hilbert space $\overline{\mathcal{H}}$, we project onto the **gauge-invariant** subspace in which the redundancy is removed. Since applying $\mathcal{G}_j$ to a site cuts the number of degrees of freedom in half, applying G brings our extended Hilbert space from 2^{2n} dimensions back down to 2^n as we had originally. We denote the gauge-invariant subspace by:

$$\mathcal{H}_{\text{gauged}} := \mathcal{G}\overline{\mathcal{H}} \tag{3.42}$$

This process of extending a Hilbert space by coupling to background defects and then enforcing a Gauss law on each site is known more generally as **gauging**.

Now, using the fact that

$$\tilde{\sigma}^x_{j-\frac{1}{2}}\sigma^x_j\,\tilde{\sigma}^x_{j+\frac{1}{2}} = 1 \tag{3.43}$$

in $\mathcal{H}_{\text{gauged}}$, we can rewrite our extended Hamiltonian from Eq. 3.36 as:

$$-\sum_{i=1}^{n}\sigma^z_i\tilde{\sigma}^z_{i+\frac{1}{2}}\sigma^z_{i+1} - h\sum_{i=1}^{n}\tilde{\sigma}^x_{i-\frac{1}{2}}\tilde{\sigma}^x_{i+\frac{1}{2}} \tag{3.44}$$

Observe that now the two terms in our Hamiltonian have seemingly swapped place. The magnetic field term is now supported on two adjacent dual sites while the interaction term is only supported on a single link and the two sites it's attached to. We may wonder if there is some basis change we can apply to make this seem more obvious. Naturally, the first thing we should do is map $\sigma^x_{i-\frac{1}{2}}\sigma^x_{i+\frac{1}{2}} \mapsto \sigma^z_{i-\frac{1}{2}}\sigma^z_{i+\frac{1}{2}}$. We can do this by acting conjugating each site simultaneously with the *Hadamard matrix*:

$$H_{j+\frac{1}{2}} = \frac{1}{\sqrt{2}}\left(\sigma^z_{j+\frac{1}{2}} + \sigma^x_{j+\frac{1}{2}}\right) \tag{3.45}$$

It is straightforward to see that acting with a Hadamard on every link maps $\sigma^x_{j+\frac{1}{2}} \leftrightarrow \sigma^x_{j+\frac{1}{2}}$, which means our Hamiltonian transforms as:

$$\left(\prod_{j=1}^{n}H_{j+\frac{1}{2}}\right)H_{\text{gauged}}\left(\prod_{j=1}^{n}H_{j+\frac{1}{2}}\right) = -\sum_{i=1}^{n}\sigma^z_i\tilde{\sigma}^x_{i+\frac{1}{2}}\sigma^z_{i+1} - h\sum_{i=1}^{n}\tilde{\sigma}^z_{i-\frac{1}{2}}\tilde{\sigma}^z_{i+\frac{1}{2}} \tag{3.46}$$

Now, we want to see if we can express each $\sigma^z_i\tilde{\sigma}^x_{i+\frac{1}{2}}\sigma^z_{i+1}$ term as an operator supported on a single link $i+\frac{1}{2}$. First, we define the following operator which is supported on two sites i and j:

$$\text{CZ}_{i,j} = \frac{1}{2}\left(1 + \sigma^z_i + \sigma^z_j - \sigma^z_i\sigma^z_j\right) \tag{3.47}$$

We call this operator the controlled Z operator, as if the i'th spin is in the $|+\rangle$ state, it does nothing to the j'th state, where as if the i'th state is $|-\rangle$, it acts on j with σ^z_j.

It is fairly straight forward to check by hand that conjugating by $\text{CZ}_{j+\frac{1}{2},\,j+1}$ maps

$$\sigma^z_j\tilde{\sigma}^x_{j+\frac{1}{2}}\sigma^z_{j+1} \mapsto \sigma^z_j\tilde{\sigma}^x_{j+\frac{1}{2}} \tag{3.48}$$

for all j. The intuition here is that the first $\text{CZ}_{j+\frac{1}{2},\,j+1}$ will act, and then $\sigma^x_{j+\frac{1}{2}}$ will swap the control qubit so that the second $\text{CZ}_{j+\frac{1}{2},\,j+1}$ will act in the opposite way. This way, there are always two σ^z_{j+1}'s which will cancel each other.

It can be found similarly that conjugating by $\text{CZ}_{j,\,j+\frac{1}{2}}$ maps $\sigma^z_j\tilde{\sigma}^x_{j+\frac{1}{2}} \mapsto \tilde{\sigma}^x_{j+\frac{1}{2}}$. If we conjugate by these two controlled Z's for every j, our Hamiltonian becomes:

$$H_{\text{gauged}} = -\sum_{i=1}^{n}\tilde{\sigma}^z_{i+\frac{1}{2}} - h\sum_{i=1}^{n}\tilde{\sigma}^x_{i-\frac{1}{2}}\tilde{\sigma}^x_{i+\frac{1}{2}} \tag{3.49}$$

which is exactly an Ising model on the dual lattice with inverse coupling, i.e Kramers-Wannier Duality!

To summarize, we found that if we gauge the Hamiltonian $H(h)$ by summing over all possible

$\mathbb{Z}_2$ defects and imposing a Gauss law and then conjugate by the unitary operator:

$$U_{\text{KW}} = \prod_j H_{j+\frac{1}{2}} \prod_j \text{CZ}_{j-\frac{1}{2},\, j} \prod_j \text{CZ}_{j,\, j+\frac{1}{2}} \tag{3.50}$$

we obtain $h\, \tilde{H}(\frac{1}{h})$, where $\tilde{H}$ is the Ising Hamiltonian on the dual lattice.

This can be implemented by an operator $\text{KW} : \mathcal{H} \to \mathcal{H}_{\text{gauged}}$ which acts as:

$$(\text{KW})\, \sigma_i^z \sigma_{i+1}^z = \tilde{\sigma}_{i+\frac{1}{2}}^x\, (\text{KW}) \qquad (\text{KW})\, \sigma_i^x = \sigma_{i-\frac{1}{2}}^z \sigma_{i+\frac{1}{2}}^z\, (\text{KW}) \tag{3.51}$$

However, if we consider our theory to be on a closed ring (i.e periodic or anti-periodic boundary conditions) then this operator is not only not unitary, but it is in fact *non-invertible*. Indeed, if it was then we would have:

$$\prod_{j=1}^n \sigma_j^x = \prod_{j=1}^n (\text{KW})^{-1}\, \sigma_{j-\frac{1}{2}}^z \sigma_{j+\frac{1}{2}}\, (\text{KW})$$

$$= (\text{KW})^{-1} \left(\prod_{j=1}^n \sigma_{j-\frac{1}{2}}^z \sigma_{j+\frac{1}{2}}^z \right) (\text{KW})$$

Since the product of all pairs of Pauli-Z operators on a closed chain will cancel to the identity, we obtain

$$\prod_{j=1}^n \sigma_j^x = 1 \tag{3.52}$$

which is a contradiction, as the spin-flip symmetry also has -1 eigenstates. The non-invertibility of the Kramers-Wannier gauging map reflects the non-local, topological nature of this duality which will be explored in the final section of this thesis. The Kramers-Wannier duality mapping (KW) can indeed be expressed as a highly non-local, but still closed form operator, from which interesting fusion relations can be derived [20], [21], [7].

3.4 Fermionization of the Ising Model

While we have done a great deal of analysis of the symmetries of the 1d transverse field Ising model, we have yet to discuss how to actually "solve" this model. In this section, we will show that this model can be mapped exactly onto a model of non-interacting particles, called fermions, from which the energy spectrum can be more easily understood. Fermions are particles which are created and annihilated from operators $c_i^\dagger$ and c_i respectively which obey the anti-commutation relations:

$$\{c_i,\, c_j^\dagger\} = \delta_{ij} \qquad \{c_i,\, c_j\} = 0 \tag{3.53}$$

A Hamiltonian is then usually constructed from these creation and annihilation operators from which it is easily found that the energy states are entirely determined by the number of fermions. The fact that the particles anticommute tells us that there can only be one particle per site (Pauli's Exclusion principle) which then makes the problem even easier.

Because of their anti-commuting nature, the Pauli matrices are naturally associated with fermions, so it is very plausible to believe we can phrase the Ising Hamiltonian in terms of fermionic creation and annihilation operators. The logical first step in expanding on this hunch would be to defining creation and annihilation operators directly from the Pauli matrices:

$$\sigma_s^{\pm} i = \frac{1}{2}(\sigma_i^z \pm i\sigma_i^y) \tag{3.54}$$

It is easy to see that these operators obey the following (anti-)commutation relations:

$$\{\sigma_i^+,\, \sigma_i^-\} = 1 \qquad [\sigma_i^+, \sigma_j^-] = 0 \tag{3.55}$$

for $i \neq j$. Thus, if we tried to write the Ising model in terms of operators like these, we would essentially have a model in which particles behave mutually as fermions when they occupy the same site but behave as bosons (commuting particles) when they are on different sites. This is strange, but given that we are working in 1 spatial dimension, perhaps we should expect such strangeness.

In one spatial dimension, there is not enough room for particles to exchange positions without passing through one another. But since these particles behave as fermions when they are on the same site, Pauli's exclusion principle can be applied and intuitively these particles will never be able to cross paths and hence will never be able to exchange positions. Since these particles can never actually exchange positions, it is not crazy to assume that in one dimension, particles that are fermions on the same site but are bosons on different sites may as well simply be fermions altogether.

Of course this argument is not at all rigorous, but Jordan and Wigner indeed found a way to show the relation between the 1d quantum Ising model and a system of free fermions in 1928 [22]. Considering our lattice to have periodic boundary conditions, we begin by defining new creation operators which create kinks that extend to various sites in the chain, similar to the μ_i^z operators from a couple setions ago:

$$f_i^\dagger := \left(\prod_{0<j<i} \sigma_j^x\right)\sigma_i^- \qquad f_i = \left(\prod_{0<j<i} \sigma_j^x\right)\sigma_i^+ \tag{3.56}$$

It is a very straight forward computation similar to what we have been doing in the past few sections to see that these new operators obey the following anti-commutation relations:

$$\{f_i,\, f_j^\dagger\} = \delta_{ij} \qquad \{f_i,\, f_j\} = 0 \tag{3.57}$$

From here, we can observe that:

$$\begin{aligned}(f_i - f_i^\dagger)(f_{i+1} + f_{i+1}^\dagger) &= (\sigma_i^+ - \sigma_i^-)\sigma_i^x(\sigma_{i+1}^+ + \sigma_{i+1}^-) \\ &= i\sigma_i^y\sigma_i^x\sigma_{i+1}^z \\ &= \sigma_i^z\sigma_{i+1}^z\end{aligned} \tag{3.58}$$

We also have that:

$$\begin{aligned} 2f_i^\dagger f_i - 1 &= 2\sigma^-\sigma^+ - 1 \\ &= \frac{1}{2}(\sigma_i^z - i\sigma_i^y)(\sigma_i^z + i\sigma_i^y) - 1 \\ &= (1 + i\sigma_i^z\sigma_i^y) - 1 \\ &= \sigma_i^x \end{aligned} \tag{3.59}$$

Thus, we can write the Ising Hamiltonian in terms of our creation and annihilation operators via:

$$H = -\sum_{i=1}^{n}(f_i - f_i^\dagger)(f_{i+1} + f_{i+1}^\dagger) - h\sum_{i=1}^{n}(2f_i^\dagger f_i - 1) \tag{3.60}$$

Observe that the product of the creation and annihilation operator $f_i^\dagger f_i$ has eigenvalues 1 and 0, which we interpret as counting whether or not site i has a fermion on it. With this in mind, we define the total particle number operator to be:

$$N := \sum_{i=1}^{n} f_i^\dagger f_i = \sum_{i=1}^{n} \frac{(1 + \sigma_i^x)}{2} \tag{3.61}$$

From here, it is easy to see that:

$$[N, H] \neq 0 \tag{3.62}$$

which means that the total number of fermions is not conserved as the system evolves in time. This is an issue if we plan to find the energy spectrum of the Ising model by treating kinks in the chain as particles with discrete energy values.

To combat the issue of diagonalization, we will perform a discrete Fourier transform on our creation operators to take them to **momentum space**. Specifically, denoting the position at a site j by $x_j := ja$ where a is the lattice spacing, we write:

$$f_k = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} f_j e^{ikx_j} \tag{3.63}$$

where the momentum can take on the following values $k = 0, \frac{2\pi}{na}, \cdots, \frac{2\pi(n-1)}{na}$. Taking the inverse Fourier transform of Eq. 3.63 and plugging into the Hamiltonian in Eq. 3.60, we find:

$$\begin{aligned} H &= \sum_{i=1}^{n}(f_i^\dagger f_{i+1} + f_{i+1}^\dagger f_i) + \sum_{i=1}^{n}(f_{i+1}f_i + f_i^\dagger f_{i+1}^\dagger) - h\sum_{i=1}^{n}(2f_i^\dagger f_i - 1) \\ &= \sum_k f_k^\dagger f_k(e^{ika} + e^{-ika}) - \sum_k (e^{ika} f_k f_{-k} + e^{ika} f_k^\dagger f_{-k}^\dagger) - h\sum_k (2f_k^\dagger f_k - 1) \\ &= \sum_k \Big(2f_k^\dagger f_k(\cos ka - h) - (e^{ika} f_k f_{-k} + e^{ika} f_k^\dagger f_{-k}^\dagger) + h\Big) \end{aligned} \tag{3.64}$$

From here, observe that we can symmetrize the second term to obtain:

$$\begin{aligned}\sum_k e^{ika} f_k f_{-k} &= \frac{1}{2}\left(\sum_k e^{ika} f_k f_{-k} + \sum_k e^{-ika} f_{-k} f_k\right) \\ &= \frac{1}{2}\left(\sum_k e^{ika} f_k f_{-k} - \sum_k e^{-ika} f_k f_{-k}\right) \\ &= \sum_k i \sin(ka) f_k f_{-k}\end{aligned} \tag{3.65}$$

where we used the fact that $\{f_k,\ f_{k'} = 0\}$, which can be easily derived from Eq.3.63.

Plugging this expression into Eq. 3.64, we obtain:

$$H = \sum_k (2 f_k^\dagger f_k (\cos ka - h) - i \sin(ka)(f_k f_{-k} + f_k^\dagger f_{-k}^\dagger) + h) \tag{3.66}$$

Now the Hamiltonian is in a form that will make it easy for us to diagonalize. Before we do so, we want to remedy the fact that the number of fermions created by $f^\dagger$ is not conserved. To do so, we will use a method called the Bogoliubov transformation [23]. The idea is that we want to find a new set of "coordinates" for our creation and annihilation operators such that the terms like $f_k f_{-k}$ and $f_k^\dagger f_{-k}^\dagger$ vanish. To perform our coordinate change, we define a new operator:

$$\gamma_k = u_k f_k - i v_k f_{-k}^\dagger \tag{3.67}$$

for some coefficients $u_k, v_k \in \mathbb{R}$. We want $\gamma_k^\dagger$ and γ_k to be fermionic creation and annihilation operators, so we should choose u_k and v_k such that:

$$\{\gamma_k,\ \gamma_l^\dagger\} = \delta_{kl} \qquad \{\gamma_k,\ \gamma_l\} = 0 \tag{3.68}$$

It is easy to check that this places the following constraints on u_k and v_k:

$$u_k^2 + v_k^2 = 1 \qquad u_k = u_{-k} \qquad v_k = -v_{-k} \tag{3.69}$$

This is satisfied if u_k and v_k are expressed in terms of trigonometric functions as:

$$u_k = \cos\frac{\theta_k}{2} \qquad v_k = \sin\frac{\theta_k}{2} \tag{3.70}$$

for some θ_k. Now, after a bit of algebra, it can be found that the expression for γ_k can be inverted to give:

$$f_k = u_k \gamma_k + i v_k \gamma_{-k}^\dagger \tag{3.71}$$

Plugging Eq. 3.71 into the expression for the Hamiltonian we found in Eq. 3.66, we find the following after a good bit of simplification:

$$\begin{aligned}H = \sum_k \Big(&\left(A(u_k^2 - v_k^2) + 4B u_k v_k\right) \gamma_k^\dagger \gamma_k + i\left(A u_k v_k - B(u_k^2 - v_k^2)\right)\left(\gamma_k \gamma_{-k} + \gamma_k^\dagger \gamma_{-k}^\dagger\right) \\ &+ h + A v_k^2 - 2B u_k v_k\Big)\end{aligned} \tag{3.72}$$

where $A := 2(\cos ka - h)$ and $B := \sin ka$. Now, we'd like to choose θ_k, as in Eq. 3.70, such that the non-diagonal term vanishes. For this to happen, we need:

$$Au_k v_k - B\left(u_k^2 - v_k^2\right) = A\cos\frac{\theta_k}{2}\sin\frac{\theta_k}{2} - B(\cos^2\frac{\theta_k}{2} - \sin^2\frac{\theta_k}{2}) = 0 \tag{3.73}$$

From basic trig identities, we find that this implies that θ_k must satisfy:

$$\tan\theta_k = \frac{\sin ka}{\cos ka - h} \tag{3.74}$$

With this choice of θ_k, we find after some straight forward, but tedious, computation that our Hamiltonian becomes:

$$H = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right) \tag{3.75}$$

where

$$\epsilon_k = 2\sqrt{1 - 2h\cos ka + h^2} \geq 0 \tag{3.76}$$

Observe that this Hamiltonian has the following *commutation* relations with the creation and annihilation operators for all k:

$$[H, \gamma_k^\dagger] = \epsilon_k \gamma_k^\dagger \qquad [H, \gamma_k] = -\epsilon_k \gamma_k \tag{3.77}$$

Thus, any ground state $|\Psi_0\rangle$ with energy E_0 must obey $\gamma_k |\Psi_0\rangle = 0$ for all k, as otherwise:

$$H\gamma_k |\Psi_0\rangle = (E_0 - \epsilon_k)\gamma_k |\Psi_0\rangle \tag{3.78}$$

meaning $\gamma_k |\Psi_0\rangle$ has a lower energy than $|\Psi_0\rangle$, which is a contradiction.

This means that the ground state energy of the 1d transverse field Ising model is:

$$E_0 = -\frac{1}{2}\sum_k \epsilon_k \tag{3.79}$$

and the possible eigenstates of our Hamiltonian are simply given by every way of acting with distinct fermionic creation operators on the ground state(s):

$$\left\{ (\gamma_{k_1}^\dagger)^{i_1} \cdots (\gamma_{k_n}^\dagger)^{i_n} |\Psi_0\rangle \,\middle|\, i_1, \ldots, i_n \in \{0,1\},\ H|\Psi_0\rangle = E_0 |\Psi_0\rangle \right\} \tag{3.80}$$

where we have to take into account that for $h \ll 1$ the ground state is doubly degenerate, as we found in a previous section.

Thus, the entire energy spectrum can be solved exactly:

$$\mathrm{Spec}(H) = \left\{ E_0 + \sum_k i_k \epsilon_k \,\middle|\, i_k \in \{0,1\}\ \forall\, k \right\} \tag{3.81}$$

From the ground state energy, an exact expression for the free energy of the 2d classical Ising model in the thermodynamic limit can be found, meaning that we can **finally** exactly solve the 2d Ising model.

3.5 A Peek into Conformal Field Theory

To summarize, the previous section, we began by performing a Jordan-Wigner transformation to express Ising spins in terms of non-interacting fermions living on the sites of the chain. Then, we took the discrete Fourier transform of the Hamiltonian to instead consider particles in terms of their momenta $k = \frac{2\pi j}{na}$ for lattice sites $j = 0, 1, \cdots, n-1$. Lastly, using a Bogoliubov transformation, we mapped the 1d quantum Ising model into the following Hamiltonian:

$$H = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \tag{3.82}$$

where $\epsilon_k = 2\sqrt{1 - 2h\cos ka + h^2} \geq 0$ is the energy of a single fermion of momentum k. We refer to the lowest possible energy of a fermion as the *gap* of the system. It is easy to see from the expression for ϵ_k that its lowest possible value is for a fermion of momentum $k = 0$, in which case its energy is:

$$\epsilon_{\min} = \epsilon_0 = 2\sqrt{h^2 - 2h + 1} = 2|h - 1| \tag{3.83}$$

Curiously, at $h = 1$, the system becomes *gapless*, meaning that for large enough n, there exists particles of arbitrarily low energy which will completely dominate the behavior of the system at low temperatures. One may wonder if $h = 1$ is then the point at which the model undergoes a phase transition from the ordered phase with the degenerate ground state to the disordered phase with the unique ground state. Indeed, from our expression for the partition function in terms of the ground state energy in Eq. 3.4, we find that we can identify the free energy density with the ground state energy density, giving the following expression:

$$\begin{aligned} f &= -\frac{1}{2} \lim_{n\to\infty} \frac{1}{na} \sum_{j=0}^{n} \epsilon_{\frac{2\pi j}{na}} \\ &= -\frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \epsilon_k \end{aligned} \tag{3.84}$$

From here, as was found by Onsager all the way back in the 40s through a different method, it can be found that the second derivative of f has a singularity at $h = 1$, finally proving that there is a phase transition at $h = 1$.

Not only is there a phase transition, but recall from Eq. 3.6 that the correlation length of the Ising model is given by:

$$\xi = \frac{1}{E_1 - E_0} \tag{3.85}$$

where E_1 is the energy of the first excited state. But since at $h = 1$ the system is gapless, we have $\xi \to \infty$, essentially meaning that changes in spins will propagate infinitely far. This interesting behavior leads us to say that the phase transition at $h = 1$ is *critical*.

Because of the infinite correlation length, there is no intrinsic length scale of our system – the microscopic details of the system smooth over and the lattice spacing becomes irrelevant. At $h = 1$, we can zoom out indefinitely, taking the lattice spacing to be 0 in the spatial direction as well while leaving all of the interactions in the system intact and ensuring that

all important physical quantities remain finite.

We now consider the continuum limit of this theory near the critical point $h = 1$. For small $ka \ll 1$, we expand the cosine:

$$\cos(ka) \approx 1 - \frac{1}{2}(ka)^2,$$

which gives

$$\epsilon_k \approx 2\sqrt{(1-h)^2 + h(ka)^2}. \tag{3.86}$$

At the critical point $h = 1$, this becomes a linear dispersion relation:

$$\epsilon_k \approx 2|k|a. \tag{3.87}$$

Thus, the low-energy excitations are relativistic, with velocity $v = 2a$. Taking the continuum limit involves sending the lattice spacing $a \to 0$, while keeping the total length $L = na$ fixed. The momentum becomes a continuous variable $k \in \mathbb{R}$, and the sum turns into an integral:

$$H \to \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi}\, \epsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right), \tag{3.88}$$

where $\Lambda \sim \pi/a$ is a UV cutoff set by the finite size of the lattice.

At $h = 1$, we substitute $\epsilon_k = 2|k|a$:

$$H = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi}\, 2|k|a \left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right). \tag{3.89}$$

We now rescale the fermion operators to define a continuum field:

$$\psi_k := \sqrt{a}\, \gamma_k, \tag{3.90}$$

so that the canonical anticommutation relations remain finite as $a \to 0$. The Hamiltonian becomes:

$$H = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, v|k| \left(\psi_k^\dagger \psi_k - \frac{1}{2}\right), \quad \text{with } v = 2a. \tag{3.91}$$

To Fourier transform back to position space, we define the continuum field

$$\psi(x) := \int \frac{dk}{2\pi} \left(\gamma_k e^{ikx} + \gamma_k^\dagger e^{-ikx}\right). \tag{3.92}$$

Although the Bogoliubov fermions γ_k and $\gamma_k^\dagger$ appear to be complex, recall that they are in fact related by

$$\gamma_k^\dagger = \gamma_{-k}, \tag{3.93}$$

which follows from the structure of the Bogoliubov transformation:

$$\gamma_k = u_k f_k - i v_k f_{-k}^\dagger, \quad \gamma_k^\dagger = u_k f_k^\dagger + i v_k f_{-k}, \tag{3.94}$$

together with the symmetry conditions $u_k = u_{-k}$ and $v_k = -v_{-k}$, which were imposed to ensure fermionic statistics and diagonalization of the Hamiltonian.

Using this identity, the field $\psi(x)$ becomes manifestly real:

$$\psi(x) = \int \frac{dk}{2\pi} \left(\gamma_k e^{ikx} + \gamma_{-k} e^{-ikx}\right) = \psi(x)^\dagger. \tag{3.95}$$

Thus, the field $\psi(x)$ satisfies a reality condition and describes a **Majorana fermion** in the continuum limit.

In terms of this field, the Hamiltonian becomes:

$$H = \frac{i}{2} \int dx\, \psi(x) \partial_x \psi(x), \tag{3.96}$$

which is the Hamiltonian of a massless Majorana fermion in $1+1$ dimensions. This theory is a conformal field theory with central charge $c = \frac{1}{2}$. The criticality of the Ising model at $h = 1$ manifests naturally in this continuum limit as invariance under local conformal transformations. In two dimensions, this symmetry is generated by an infinite-dimensional algebra which places strong constraints on all of the correlation functions, allowing the theory to be solved exactly!

In conformal field theory, observables are represented by local operators, and among these, a special class known as **primary operators** play a fundamental role. These are local fields $\mathcal{O}(z, \bar{z})$ that transform covariantly under conformal transformations such as to leave their correlation functions invariant:

$$\mathcal{O}(z, \bar{z}) \mapsto \left(\frac{df}{dz}\right)^{-h} \left(\frac{d\bar{f}}{d\bar{z}}\right)^{-\bar{h}} \mathcal{O}(f(z), \bar{f}(\bar{z})), \tag{3.97}$$

where h and $\bar{h}$ are called the *conformal weights* of the operator. Primary operators are annihilated by the positive modes of a central extension of the 2d conformal algebra, known as the *Virasoro algebra*, and serve as the highest-weight states in the representation theory of the conformal symmetry. All other local fields in the theory can be built as derivatives or products of these primaries, known as *descendants*.

In the Ising conformal field theory, which describes the critical point ($h = 1$) of the 2d classical Ising model and the massless Majorana fermion, there are only three primary fields:

Operator	**Name**	**Conformal Weights**
$\mathbb{1}$	Identity	$(0, 0)$
$\varepsilon(z, \bar{z})$	Energy density	$\left(\frac{1}{2}, \frac{1}{2}\right)$
$\sigma(z, \bar{z})$	Spin field	$\left(\frac{1}{16}, \frac{1}{16}\right)$

Table 3.1: Primary fields in the critical 2D Ising CFT and their conformal weights.

These fields obey nontrivial fusion rules, such as:

$$\sigma \times \sigma = \mathbb{1} + \varepsilon, \quad \varepsilon \times \varepsilon = \mathbb{1}, \quad \sigma \times \varepsilon = \sigma. \tag{3.98}$$

These rules reflect the possible channels through which operator insertions can fuse, and they encode the underlying algebraic structure of the theory. The expression of correlation functions in terms of fusion rules naturally leads one to reinterpret them in terms of topological defect lines, known as *Verlinde lines*—perhaps the first motivating example of topological defects in the sense considered in this thesis [24]. We will see in the final chapter that the structure of these defects is closely related to the topological defects we encountered in the 1d quantum Ising chain, and that they naturally arise as manifestations of the $\mathbb{Z}_2$ global symmetry and Kramers-Wannier duality in the 2d classical Ising model. From this perspective, we argue that the solvability of conformal field theories is not merely a consequence of conformal symmetry, but rather stems from something more fundamental: the fact that in two dimensions, such theories admit a rich algebraic structure of topological defect lines, inherited from the symmetries of the underlying lattice model.

Chapter 4

Fusion Categories and Topological Defects

Whether they are introduced to explain Kramers-Wannier duality in terms of gauging the global $\mathbb{Z}_2$ symmetry in the 1d quantum Ising model or you take the continuum limit and introduce them to understand the fusion of primary operators in the Ising CFT, topological defect lines are crucial to understanding the nuances of physical systems. Therefore, understanding the mathematical structure of these defects is vital to understanding the global symmetries of even the most basic physical models. It turns out that the most natural mathematical language in which we can understand the essence of these defects is **category theory**.

4.1 Why Category Theory?

You have likely heard that category theory is just abstract nonsense – so what could it possibly have to do with physics? Before we answer this, let's define what exactly a category is.

Definition 4.1.1

A *(small) category* $\mathcal{C}$ consists of:

1. A set of *objects*, called $\mathrm{Ob}(\mathcal{C})$; we typically refer to an object by simply $x \in \mathcal{C}$.
2. For each pair of objects x, y, a set of *morphisms* $\mathrm{Hom}(x, y)$.
3. A *composition rule*: for any $f \in \mathrm{Hom}(x, y)$ and $g \in \mathrm{Hom}(y, z)$, there is a composite morphism $g \circ f \in \mathrm{Hom}(x, z)$.

This data satisfies:

- **Associativity:** $(h \circ g) \circ f = h \circ (g \circ f)$ whenever the compositions are defined.
- **Identity:** For each object x, there exists an identity morphism $\mathrm{id}_x \in \mathrm{Hom}(x, x)$ such that $\mathrm{id}_x \circ f = f$ and $g \circ \mathrm{id}_x = g$ for all appropriate f, g.

Some of the classic, motivating examples of categories for the working mathematician are

as follows:

- **Set:** The category whose objects are sets and morphisms are functions between sets.
- **Grp:** The category whose objects are groups and morphisms are homomorphisms between groups.
- $\mathbf{Vec}_{\mathbb{C}}$: The category whose objects are finite-dimensional vector spaces over $\mathbb{C}$ and whose morphisms are $\mathbb{C}$-linear maps.

However, these categories are not very instructive for understanding why category theory may be useful for a practicing physicist. A more intuitive example is the following:

Example.

Consider the category **Tangle**, whose objects are finite collections of oriented points in $\mathbb{R}^3$ and whose morphisms are given by arrows that connect two collections of points together according to the orientation of the points. For instance, a morphism from $(+-+--+)$ to $(--+-)$ is given below:

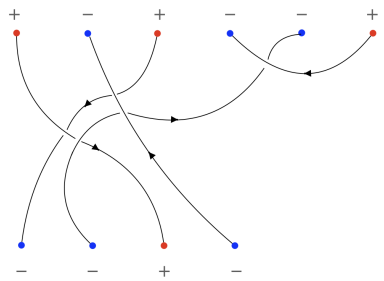


Figure 4.1: Morphism from $(+-+--+)$ to $(--+-)$

Now, an associative composition is defined on morphisms by stacking two diagrams on top of each other, and the identity morphism from an object to itself simply consists of straight lines connecting each point to its copy. We kill two birds with one stone and show the composition of a morphism from $(+-)$ to $(-+)$ with the identity map of $(-+)$ to itself below:

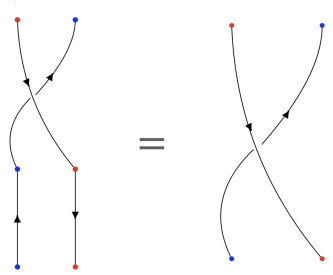


Figure 4.2: Composition of morphism with identity

Expanding upon the importance of morphisms, we should define a notion of a structure-

preserving map between two categories.

Definition 4.1.2

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

- An assignment to each object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$,
- An assignment to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$,

such that the following conditions hold:

1. (**Preservation of identities**) For every object X in $\mathcal{C}$,
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$
2. (**Preservation of composition**) For all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$,
$$F(g \circ f) = F(g) \circ F(f).$$

Category theory provides a natural language for describing topological defects, since it is not particularly meaningful to consider a defect at a single site. The essence of topological defects lies in how they can be deformed with local operators in ways that preserve the partition function. The topological defects in a $(1+1)$-d system naturally form a category with objects as equivalence classes of defects and morphisms coming from local operators which either map a defect into itself or potentially into another kind of defect.

However, a plain-old category is not enough to encapsulate all of the important properties of topological defects. To start, we know that we need some way of summing over multiple defects on a single site as we did during our gauging procedure. We also know that we should be able to fuse two defects together to obtain a new one; it would also be nice to have some way of creating a pair of defects from the vacuum or annihilating two defects as we had with the $\mathbb{Z}_2$ defects. With these needs in mind, we will try and enrich our original definition of a category with new structures.

4.2 What is a 2-Vector Space?

In this section we will define an additive structure for categories and see how it can be viewed as a higher-dimensional generalization of vector space. No one denies the usefulness of linear algebra in physics, so as we generalize our physical theories to have higher dimensional symmetry operators, it only makes sense that we should have a higher version of our usual linear algebra as well.

Before we do anything else, we want to formalize the notion of two objects being equivalent in a category. To do so, we say that a morphism $f \in \mathrm{Hom}(x, y)$ as an **isomorphism** if there exists another morphism $g \in \mathrm{Hom}(y, x)$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$. If there exists an isomorphism between two objects a and b we say that a and b are isomorphic, or $a \simeq b$.

With this out of the way, we know that in a category of topological defects, morphisms should be given by linear operators acting on a Hilbert space. However, presently, we have

no notion of taking linear combinations of morphisms. To account for this, we define the following notion:

Definition 4.2.1

Definition. A *linear category* (over $\mathbb{C}$) is a category $\mathcal{C}$ such that for every pair of objects $x, y \in \mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a vector space over $\mathbb{C}$, and composition of morphisms is bilinear in both arguments.

A *linear functor* between linear categories is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that for all morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ and all scalars $\alpha, \beta \in \mathbb{C}$,

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g).$$

An easy example of a linear category is the category $\mathrm{Vec}_{\mathbb{C}}$, as the set of linear maps between two vector spaces naturally forms a vector space. We also know that we can take direct sums of vector spaces; perhaps we can try to generalize this property to other categories so that we can utilize it in our description of topological defects.

Let's think about what makes $\mathrm{Vec}_{\mathbb{C}}$ so special – why can we take direct sums in this category but not in any generic category, and what are the essential properties of the direct sum? Well, for starters, if V and W are vector spaces, then there are canonical embeddings $i_V : V \hookrightarrow V \oplus W$ and $i_W : W \hookrightarrow V \oplus W$, as well as canonical projections $\pi_V : V \oplus W \longrightarrow V$ and $\pi_V : V \oplus W \longrightarrow W$. Since V and W do not mix with each other inside of their direct sum, it's clear that these embeddings and projections will clearly obey:

$$\pi_V \circ i_V = \mathrm{id}_V \qquad \pi_W \circ i_W \qquad \pi_W \circ i_V = 0 \qquad \pi_V \circ i_W = 0 \tag{4.1}$$

We also know that we can reconstruct the direct sum from the information we get from the projections, meaning:

$$i_V \circ \pi_V + i_w \circ \pi_W = \mathrm{id}_{V \oplus W} \tag{4.2}$$

Since nothing in these conditions is specific to vector spaces, these properties can be generalized to an arbitrary linear category:

Definition 4.2.2

Given a finite collection of objects $c_1, \cdots, c_n \in \mathcal{C}$, we say that an object $c = c_1 \oplus \cdots \oplus c_n$ is a *direct sum* if there exists morphisms $i_j : c_j \to c$ and $\pi_j : c \to c_j$ that obey the following conditions:

1. $\pi_i \circ i_j = \delta_{ij} \mathrm{id}_{c_j}$ for all i and j
2. $\sum_{j=1}^{n} i_j \circ \pi_j = \mathrm{id}_c$ for all j.

It is important to note that we can write any morphism $f : c \to d$ between direct sums

$c := \bigoplus_{i=1}^{n} c_i$ and $d := \bigoplus_{i=1}^{m} d_i$ as a matrix equation:

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \pi_1 \circ f \circ i_1 & \cdots & \pi_1 \circ f \circ i_n \\ \vdots & \ddots & \vdots \\ \pi_m \circ f \circ i_1 & \cdots & \pi_m \circ f \circ i_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{4.3}$$

This gives a canonical isomorphism of vector spaces

$$\mathrm{Hom}(\bigoplus_{i=1}^{n} c_i, \bigoplus_{j=1}^{m} d_j) \longrightarrow \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathrm{Hom}(c_i, d_j) \tag{4.4}$$

where the $\bigoplus$'s on the right hand side are simply the direct sums of vector spaces.

Another aspect of vector spaces that we'd like to replicate in other categories is the fact that every vector space can be broken up into the direct sum of smaller vector spaces. There is also the trivial vector space $\{0\}$ that acts as a sort of additive identity for the direct sum. One clever way of expressing these properties in terms of morphisms in a generic category is to define the following notion:

Definition 4.2.3

Let $x \in \mathcal{C}$ and $e \in \mathrm{Hom}(x, x)$. We say that e is an *idempotent* if $e \circ e = 0$. Furthermore, we say that e *splits* if there exists an object $y \in \mathcal{C}$, which we call the image of e, with morphisms $r : x \to y$ and $s : y \to x$ such that $r \circ s = id_y$ and $s \circ r = e$. If every idempotent splits, we say that $\mathcal{C}$ is *idempotent complete*. Furthermore, if every collection of objects in $\mathcal{C}$ admits a direct sum on top of this, we say that $\mathcal{C}$ is *Cauchy complete*.

In the above definition, we think of r as a projection from a bigger object x to a smaller object y and we think of $s : y \hookrightarrow x$ as an embedding. It is a non-trivial but simple exercise to show that both the direct sum of a collection of objects and the image of an idempotent are unique up to isomorphism. Thus, Cauchy completeness is a condition that gives us many of the important properties that we wanted to replicate from vector spaces.

Remark.

Cauchy completeness is a very nice property, not only because it lets us break apart objects and build new ones, but it also gives us an additive identity 0 such that $a \oplus 0 \simeq a$ for all $a \in \mathcal{C}$, for free.

Proof. Indeed, since $0 \in \mathrm{Hom}(x, x)$ is an idempotent for any object x, in a Cauchy complete category there must exist an object y along with morphisms $r : x \to y$ and $s : y \to x$ such that $r \circ s = \mathrm{id}_y$ and $s \circ r = 0$. It is easy to see from here that this implies $\mathrm{id}_y = 0$, from which it follows that $\mathrm{Hom}(y, y) = 0$. Then, for every object z we have an object $y \oplus z$ equipped with the appropriate morphisms, but since all morphisms from y to y are zero, we have $i_z \circ \pi_z + 0 = id_{y \oplus z}$ this implies that i_z is an isomorphism from z to $y \oplus z$. Thus, y is an additive identity for $\oplus$. □

It isn't hard to see that y is unique up to isomorphism, as if y' is another zero, we have $y' \simeq y \oplus y' \simeq y$. We thus can refer to y unambiguously as the zero object, 0.

For any non-negative integer N and an object $c \in \mathcal{C}$, we denote $Nc := c \oplus \cdots \oplus c$ as simply adding c with itself N times. Now that we have some way of adding together objects and scaling them by integers, it would be nice if we could have some notion of a basis from which we can construct any other object in the category. The objects in this basis should not be able to decomposed any further and they should be easily distinguishable from each other: rmkb We say an object $c \in \mathcal{C}$ is *simple* if $\dim(\mathrm{Hom}(c, c)) = 1$. We denote the set of all simple objects in the category as $\mathrm{Irr}(\mathcal{C})$ in analogy with the set of irreducible representations of a group. Furthermore, we say that two objects $x, y \in \mathcal{C}$ are *distinct* if $\mathrm{Hom}(x, y) = \{0\}$. These definitions make sense in the context of topological defects, as we know that there are defects that cannot be deformed into one another via local operators; for example, the trivial defect and a $\mathbb{Z}_2$ defect on a single link.

Definition 4.2.4

We say that a Cauchy complete category $\mathcal{C}$ is *finitely semisimple* if there exists a finite collection of pairwise distinct, simple objects $c_1, \cdots, c_n \in \mathcal{C}$ such that for any other object $c \in \mathcal{C}$ there exists non-negative integers $N_1, \cdots, N_n$ so that $c \simeq \bigoplus_{i=1}^{n} N_i c_i$.

Remark.

This definition of semisimplicity immediately implies (and is in fact equivalent to) that any morphism is determined by where it sends the "basis vectors" or simple objects. By this, I mean that the composition of morphisms gives an isomorphism:

$$\bigoplus_{i=1}^{n} \mathrm{Hom}(a, c_i) \otimes_{\mathbb{C}} \mathrm{Hom}(c_i, b) \longrightarrow \mathrm{Hom}(a, b) \tag{4.5}$$

Notice the similarities of this expression and the usual formula for matrix multiplication in a vector space. This also solidifies the analogy of thinking of the simple elements of a category as an orthonormal basis, as if we think of $\mathrm{Hom}(-,-)$ as a sort of "inner product", the isomorphism in Eq. 4.5 is very similar to the notion of inserting a complete set of basis states

$$\langle a | \, b \rangle = \sum_i \langle a | \, c_i \rangle \, \langle c_i | \, b \rangle \tag{4.6}$$

as we are used to doing in quantum mechanics.

This motivates the following definition:

Remark.

Before proceeding, we briefly introduce two standard categorical notions that we have glossed over: the product of categories and the notion of a natural transformation between functors. These will be important for interpreting equivalences between categories and for understanding the structure-preserving properties of tensor products.

First, given two categories $\mathcal{C}$ and $\mathcal{D}$, we define their product category $\mathcal{C} \times \mathcal{D}$ as follows:

- Objects are pairs (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$.
- Morphisms are pairs $(f, g) : (c_1, d_1) \to (c_2, d_2)$ with $f \in \mathrm{Hom}(c_1, c_2)$ and $g \in \mathrm{Hom}(d_1, d_2)$.
- Composition is defined componentwise: $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1,\ g_2 \circ g_1)$.

We also define the *direct sum* of two linear categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C} \oplus \mathcal{D}$ as the linear category defined by the following:

- The underlying category of $\mathcal{C} \oplus \mathcal{D}$ is $\mathcal{C} \times \mathcal{D}$
- The linear structure on the Hom spaces is given by the direct sum of the Hom spaces in $\mathcal{C}$ and $\mathcal{D}$, i.e $\mathrm{Hom}\left((c, d),\ (c', d')\right) = \mathrm{Hom}(c, c') \oplus \mathrm{Hom}(d, d')$

Secondly, given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ is a collection of morphisms

$$\eta_c : F(c) \to G(c) \quad \text{for each } c \in \mathcal{C}$$

such that for every morphism $f : c_1 \to c_2$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(c_1) & \xrightarrow{F(f)} & F(c_2) \\ \Big\downarrow{\scriptstyle \eta_{c_1}} & & \Big\downarrow{\scriptstyle \eta_{c_2}} \\ G(c_1) & \xrightarrow[G(f)]{} & G(c_2) \end{array}$$

We say that η is a *natural isomorphism* if every component η_c is an isomorphism.

Definition 4.2.5

A *2-vector space* is a finitely semisimple, Cauchy complete, linear category.

This name goes further than mere analogy. In fact, it can be shown that any 2-vector space $\mathcal{C}$ is equivalent to the direct sum of an integer number of copies of $\mathrm{Vec}_{\mathbb{C}}$ [25]:

$$\mathcal{C} \simeq \mathrm{Vec}_{\mathbb{C}}^{\oplus n} \tag{4.7}$$

Here, by equivalence, we mean that there is a functor from $\mathcal{C}$ to $\mathrm{Vec}_{\mathbb{C}}^{\oplus n}$ with a two-sided inverse up to natural isomorphism. Since vector spaces are completely described up to isomorphism by their dimension (a non-negative integer), we can think of the integer multiplication of objects as a sort of scalar multiplication by the "ground field" $\mathrm{Vec}_{\mathbb{C}}$. Specifically, for a vector space $V \in \mathrm{Vec}_{\mathbb{C}}$ with $\dim(V) = n$, we a "scalar multiplication" of an object

$c \in \mathcal{C}$ via:

$$V \otimes c \mapsto \dim(V) \cdot c = \bigoplus_{i=1}^{n} c \tag{4.8}$$

We summarize the properties of 2-vector spaces in Table 4.1 below in relation with usual vector spaces.

2-Vector Spaces	**Vector Spaces**
$\simeq \mathrm{Vec}_{\mathbb{C}}^{\oplus n}$	$\simeq \mathbb{C}^n$
Linear categories	Vectors
Direct sum of objects	Vector addition
Multiplication by elements of $\mathrm{Vec}_{\mathbb{C}}$	Scalar multiplication by $\mathbb{C}$
Simple objects $\mathrm{Irr}(\mathcal{C})$	Basis vectors
Linear functors and morphisms	Linear maps and operators

Table 4.1: Comparison between 2-vector spaces and ordinary vector spaces

4.3 What is a Monoidal Category?

There are still a few properties that we need to enrich our category with if we want to describe topological defects mathematically. By the end of this section, we will have a fully functioning category theory with an intuitive graphical calculus that naturally describes the fusion of topological defects.

First off, we know that since the morphisms in a category of topological defects should be described by unitary operators acting on a Hilbert space, we need to incorporate some notion of taking the "dagger" of a morphism.

Definition 4.3.1

A *unitary category* is a linear category $\mathcal{C}$ equipped with a conjugate-linear map $\dagger : \mathrm{Hom}(a,b) \longrightarrow \mathrm{Hom}(b,a)$ for every $a, b \in \mathcal{C}$ that satisfies:

- $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$
- $(f^\dagger)^\dagger = f$
- $f^\dagger \circ f = 0$ if and only if $f = 0$.

for all morphisms $f : x \to y$ and $g : y \to z$ and every $x, y, z \in \mathcal{C}$.

Note that these conditions imply that $\mathrm{id}_x^\dagger = \mathrm{id}_x$ for all x. Accordingly, we define a *dagger-functor* to be a linear functor F between two dagger categories such that for every morphism f, we have:

$$F(f^\dagger) = F(f)^\dagger \tag{4.9}$$

An example of a unitary category is $\mathbf{Hilb}_{\mathbb{C}}$ – the category of finite dimensional Hilbert spaces over $\mathbb{C}$.

Now, recall from the previous section that if we had two defects on adjacent sites, we could fuse them together into form a new defect on a single site:

Thus, our category should have some notion of a fusion product in which objects can be

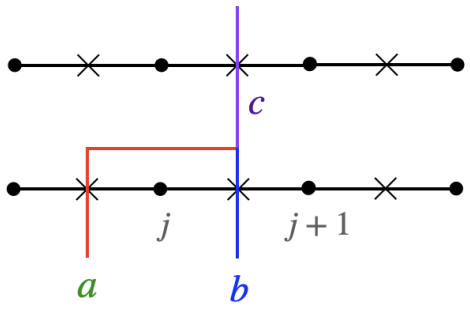


Figure 4.3: Fusion of Two Adjacent Defects

combined. Topological defects are just a generalization of particles; they are merely local excitations and deformations of the Hamiltonian of a system which can be moved around and fused together. We already have a good reference category which physicists have been using to describe how particles fuse together and split into new ones since the beginning of quantum mechanics. Namely, the category $\mathrm{Rep}(G)$ – the category of representations for a symmetry group G of a physical theory. For almost a hundred years, it has been known that the angular momentum of a particle in quantum mechanics corresponds to an irreducible representation (irrep) of SU(2), and that the combination of two particles of angular momenta j_1 and j_2 is given by tensoring the two representations and then splitting the result into a direct sum of irreps using the Clebsch-Gordan rules. Indeed, this seems like a semisimple category that is just endowed with an additional tensor product.

Example.

The category $\mathrm{Rep}(G)$ of representations of a finite group G forms a 2-vector space, as it has:

- Objects which are the representations of G
- Morphisms which are intertwiners between two representations (V, ρ) and (W, η), i.e a linear map $\varphi : V \longrightarrow W$ such that

$$\eta(g) \circ \varphi = \varphi \circ \rho(g) \tag{4.10}$$

  for all $g \in G$. Since these are linear maps, the Hom-spaces form vector spaces over $\mathbb{C}$
- The direct sum $\oplus$ is simply the direct sum of representations with the 0 object being the trivial representation $\{0\}$.
- The simple objects are the irreducible representations of G (up to isomorphism). That they are mutually distinct objects comes from the famed Schur's lemma.

Here, we specify that G is a finite group, as we want to ensure that there are a finite number of irreducible representations. This means that $\mathrm{Rep}(SU(2))$ is not a 2-vector space, but there are categories which are very closely related to it which we will see later.

Now, we want to define a type of tensor product on categories which generalizes the properties of the tensor product of representations so we can describe the fusion rules of defects.

We expect that this fusion product be somewhat associative insofar as fusing two particles and then fusing the result with another should be related in some way to if we first fused the other two particles and then fused the result with the third as can be seen in Figure 4.4.

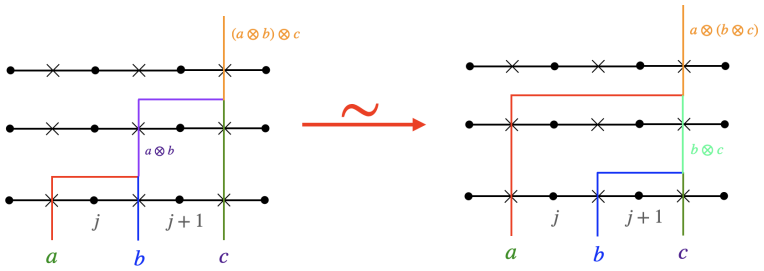


Figure 4.4: Visualization of "associativity" of topological defects

Definition 4.3.2

A *monoidal category* is a linear category $\mathcal{C}$ equipped with the following:

- A bilinear functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- A natural isomorphism α between the functors $(- \otimes -) \otimes -$ and $- \otimes (- \otimes -)$ from $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.
- A distinguished multiplicative identity element $1 \in \mathcal{C}$ along with natural isomorphisms λ and ρ with components $\lambda_c : 1 \otimes c \to c$ and $\rho_c : c \otimes 1$ for all $c \in \mathcal{C}$.

such that the pentagon and triangle diagrams in Fig. 4.5 and Fig. 4.6 commute for all $x, y, z, w \in \mathcal{C}$.

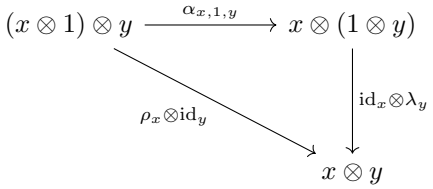


Figure 4.5: Triangle identity diagram for a monoidal category

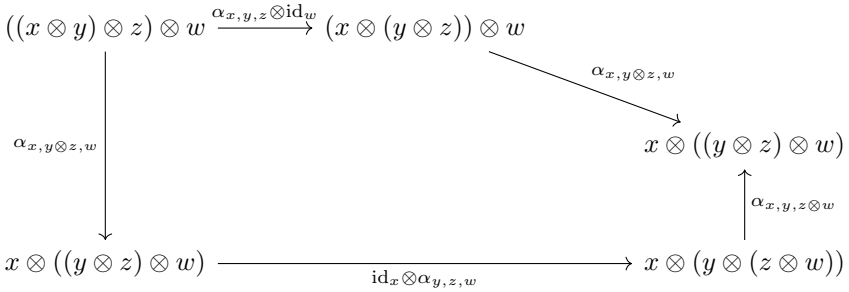


Figure 4.6: Pentagon identity diagram for associators in a monoidal category

The triangle identity simply serves to ensure that the associator and the unitors play nicely together, while the pentagon identity ensures that all possible ways of putting parentheses

around 4 objects are related to one another. Notice that $\mathrm{Vec}_{\mathbb{C}}$ and $\mathrm{Rep}(G)$ are both monoidal categories with their usual tensor products and trivial associators and unitors. There are also many more unique monoidal categories which are of use in physics:

Example.

Let G be a finite group. We define the category $\mathrm{Vec}(G)$ to be the category of G-graded vector spaces; i.e vector spaces V that can be decomposed as $V = \bigoplus_{g \in G} V_g$. $\mathrm{Vec}(G)$ has a well-defined monoidal product $V \otimes W$ whose components are given by:

$$(V \otimes W)_g = \bigoplus_{hk=g} V_h \otimes W_k \tag{4.11}$$

If we identify the identity element as $\mathbb{C}_e$, where e is the identity of G, and endow $\mathrm{Vec}(G)$ with the trivial associator and unitors from $\mathrm{Vec}_{\mathbb{C}}$, it becomes a monoidal category. However, this is not the *only* choice of associator and unitors compatible with $\mathrm{Vec}(G)$ and the tensor product in Eq. 4.11. Indeed, suppose we want to have a non-trivial associator, which acts as $\alpha_{g,h,k} = \omega(g,h,k)\mathrm{id}_{ghk}$, for some non-zero complex number $\omega(g,h,k) \in \mathbb{C}^*$. Plugging this new associator into the pentagon identity, we find that the conditions for a monoidal category still hold so as long as :

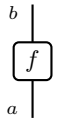
$$\omega(h,k,l)\omega(g,hk,l)\omega(g,h,k) = \omega(gh,k,l)\omega(g,h,kl) \tag{4.12}$$

for all $g,h,k,l \in G$. This is a well-known concept in group cohomology; we say that $\omega : G \times G \times G \longrightarrow \mathbb{C}^*$ must be a 3-cocycle $\omega \in Z^3(G,\, \mathbb{C}^*)$. Then, if we plug this into the triangle identity, we find that the monoidal structure is preserved if we adjust the unitors as:

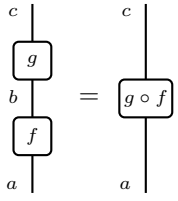
$$\lambda_g = \omega(1,1,g)^{-1}\,\mathrm{id}_g \qquad \rho_g = \omega(g,1,1)\,\mathrm{id}_g \tag{4.13}$$

We thus refer to the category of G graded vector spaces in general as $\mathrm{Vec}^{\omega}(G)$ for a 3-cocycle ω, where we say the category is twisted by ω if it is non-trivial.

We now define a graphical calculus for monoidal categories which will make future calculations much more intuitive and solidify their relation to topological defects. We represent objects as 1d strings and morphisms as 2d coupons at the interface between two strings. A morphism $f : a \to b$ is pictured below:

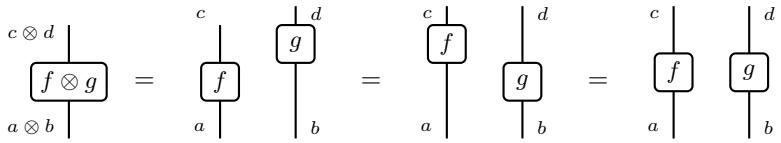


The composition of two morphisms $f : a \to b$ and $g : b \to c$ is given by the vertical stacking of the two coupons:



Because of this, we denote the identity morphism on an object by simply drawing the string with no coupon on it.

We represent the tensor product of two objects by simply placing them side by side. We can consider morphisms as coupons that are connected to both strings or split up the morphism and move them around vertically as we please.



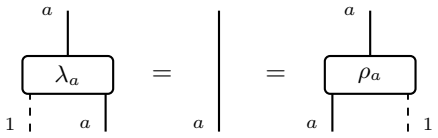
This vertical isotopy is justified by the isomorphism:

$$\mathrm{Hom}(a \otimes b,\ c \otimes d) \longrightarrow \mathrm{Hom}(a,\ c) \otimes_{\mathbb{C}} \mathrm{Hom}(b,\ d) \tag{4.14}$$

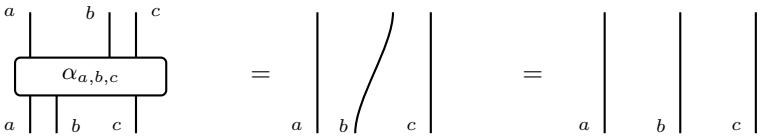
which arises naturally from the factthat $- \otimes -$ is a bilinear functor combined with the universal property of the tensor product of vector spaces. From this, we can find that for any morphisms $f : a \to c$ and $g : b \to d$, we have:

$$(\mathrm{id}_c \otimes g) \circ (f \otimes \mathrm{id}_b) = f \otimes g = (f \otimes \mathrm{id}_d) \circ (\mathrm{id}_a \otimes g) \tag{4.15}$$

We will sometimes represent the multiplicative identity, 1, by a dashed line, but oftentimes we will completely omit it and the unitors.



We do a similar thing with the associator; we will either represent it by regrouping the wires or simply omitting it entirely:



It can be proven rigorously using what is known as MacLane's coherence theorem that any equation between morphisms holds if and only if it holds in our outlined graphical calculus, up to isotopy [26]. It is not very informative to do so, so we will not include the proof.

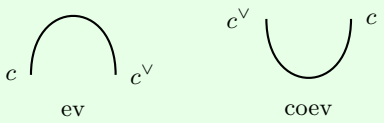
4.4 What is a Fusion Category?

The final ingredient needed to define a fully functional mathematical structure of topological defects is a notion that allows a pair of defects to be spontaneously created from the vacuum

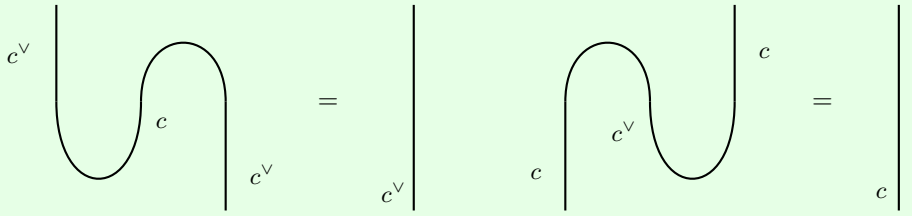
or to annihilate with each other. This should not be thought of as a multiplicative inverse, but rather as a *dual object* – in the same sense that vector spaces have duals, which provide natural maps to and from the ground field when paired together.

Definition 4.4.1

Let $\mathcal{C}$ be a monoidal category. We say that the *right dual* of an object c is an object $c^\vee$ equipped with morphisms $\mathrm{ev}_c \colon c \otimes c^\vee \to 1$ and $\mathrm{coev}_c \colon 1 \to c^\vee \otimes c$, which we denote with the following graphical calculus:



We require ev and coev to obey the snake equations:



A left dual of an object c is an object ${}^\vee c$ whose right dual is isomorphic to c, meaning $({}^\vee c)^\vee \simeq c$. If an object has both a right and left dual, we say that object is *dualizable*. Then, if every object in the category is dualizable, we say that the category is *rigid*.

This property is incredibly useful as it allows us to consider every diagram in our graphical calculus up to isotopy, leading naturally to the interpretation of objects in a monoidal category as topological defects in a physical system.

It is important to note that for a general rigid monoidal category, the left and right duals of an object are not necessarily the same. However, if the category is unitary, then if we take the dagger of both sides of each snake equations – corresponding to time reversal symmetry, or reflecting the diagrams across the horizontal axis – we find that $c^\vee$ is both the left and right dual of c. Hence, in a unitary rigid monoidal category, which we will typically be working with in a physics setting, we can speak generally of *the dual* of an object without having to differentiate between right and left.

Example.

The most obvious example of a rigid monoidal category is $\mathrm{Vec}_{\mathbb{C}}$.

- The dual of a vector space V in the category, is simply given by the usual dual vector space V^*.
- The evaluation map is given by the inner product $\langle w | v \rangle \in \mathbb{C}$
- The coevaluation map is defined by the linear map $1 \mapsto \sum_i |v_i\rangle \langle v_i|$, where $\{v_i\}$ is an ONB of V.

It is straightforward to check that these definitions satisfy the snake equations.

With this, we can finally define the structure that we have set out to find.

Definition 4.4.2

A *unitary fusion category* is a rigid, unitary monoidal 2-vector space $\mathcal{C}$ such that $1 \in \mathrm{Irr}(\mathcal{C})$.

The only extra consistency condition we need to impose is that the monoidal product and the dagger structure are compatible. That is, that $-\otimes-$ is a dagger-functor and that the associator and unitors are unitary morphisms (their dagger is their inverse).

We have discussed why all of these properties are crucial for having a mathematical theory of topological defects. We demand that the multiplicative unit 1 is a simple object, as it corresponds to the vacuum which should not be able to be decomposed exactly into non-trivial defects.

Combining semisimplicity, a monoidal product, rigidity, and unitarity facilitates a rich algebraic structure that has historically been of great use to physicists. First off, observe that for any pair of simple objects a and b, their tensor product decomposes as a direct sum of simples by semisimplicity:

$$a \otimes b = \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} N_{ab}^{c}\, c \tag{4.16}$$

for non-negative integers N_{ab}^{c}. The collection of all such coefficients for simple a, b, and c defines the *fusion rules* of the fusion category $\mathcal{C}$. These fusion rules determine the multiplication structure on the *Grothendieck ring* $K_0(\mathcal{C})$, whose elements are formal integer linear combinations of isomorphism classes of objects, with product defined by

$$[a] \cdot [b] = \sum_{c \in \mathrm{Irr}(\mathcal{C})} N_{ab}^{c}[c]. \tag{4.17}$$

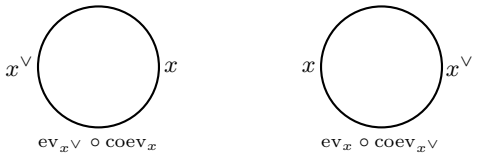
If you have any familiarity with quantum field theory, you may notice a strong resemblance between fusion rules and Feynman rules. In fact, Feynman diagrams for the Standard Model are essentially wire diagrams in our graphical calculus, but for the category $\mathrm{Rep}(P \times U(1) \times SU(2) \times SU(3))$, where P is the Poincaré group.

We will not prove it here, but there is a beautiful result known as *Ocneanu rigidity* which states that for a given collection of fusion rules, there are only finitely many possible fusion categories that can be associated to them [27]. In other words, fusion categories are very rigid objects that cannot be deformed continuously. This makes their application to physics uniquely powerful, as there is no such thing as fine-tuning parameters to make the math match up with experiment – either the math describes the real world **exactly** or it does not describe it at all!

Observe that for any simple object $a \in \mathrm{Irr}(\mathcal{C})$ and any other object $b \in \mathcal{C}$, we can endow the vector space $\mathrm{Hom}(a, b)$ with an inner product via:

$$\langle \varphi \,|\phi\rangle = \varphi^{\dagger} \circ \phi \in \mathrm{Hom}(a, a) = \mathbb{C} \tag{4.18}$$

for every φ, $\phi \in \mathrm{Hom}(a, b)$, where the last equality follows from the fact that a is a simple object. Using this, for any object x in a fusion category, we have two ways of mapping it to a positive real number: namely, $x \mapsto \mathrm{ev}_{x^\vee} \circ \mathrm{coev}_x \propto \langle \mathrm{coev}_x | \mathrm{coev}_x \rangle$ and $x \mapsto \mathrm{ev}_x \circ \mathrm{coev}_{x^\vee} \propto \langle \mathrm{coev}_{x^\vee} | \mathrm{coev}_{x^\vee} \rangle$, given in the graphical calculus by:



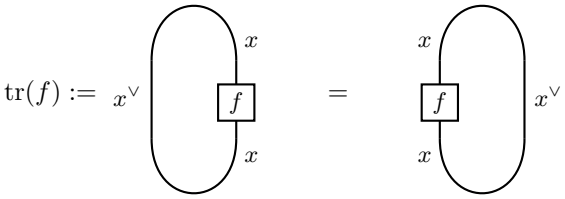
These two expressions are not inherently equal, but we have the freedom to scale $\mathrm{coev}_x \mapsto \alpha \cdot \mathrm{coev}_x$ and $\mathrm{ev}_x \mapsto \alpha^{-1} \cdot \mathrm{ev}_x$ along with $\mathrm{coev}_{x^\vee} \mapsto \beta \cdot \mathrm{coev}_{x^\vee}$ and $\mathrm{ev}_{x^\vee} \mapsto \beta^{-1} \cdot \mathrm{ev}_{x^\vee}$ for $\alpha, \beta > 0$ such that the two expressions become equal, while leaving all of the snake equations intact. If we do this for every x, we can then refer unambigously to $d_x := \mathrm{ev}_{x^\vee} \circ \mathrm{coev}_x$ as the *quantum dimension* of x. From now on, we will take this normalization condition to be included in the definitions of ev_x and coev_x for every x. We define the *dimension* of a unitary fusion category to be given by:

$$\dim \mathcal{C} = \sum_{c \in \mathrm{Irr}(\mathcal{C})} d_c^2 \tag{4.19}$$

Remark.

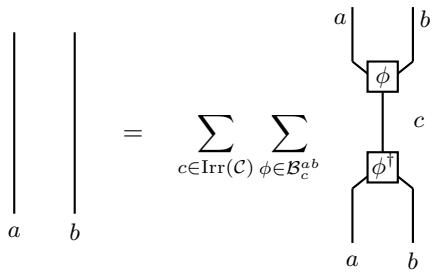
> The dimension of a simple object is sometimes defined differently. Observe that for any simple object c, its fusion rule N_{ab}^c is simply a matrix whose indices are labeled by the simple objects a and b. Since all of the entries of this matrix are manifestly non-negative, one can define the *Frobenius–Perron dimension* $\mathrm{FPdim}(c)$ as the largest positive eigenvalue of the fusion matrix N^c which is guaranteed to exist the Perron-Froebenius Theorem.
>
> This assignment extends uniquely to a ring homomorphism $\mathrm{FPdim} : K_0(\mathcal{C}) \to \mathbb{R}_{\geq 0}$ and agrees with the dimension defined categorically via the composition $\mathrm{ev}_c \circ \mathrm{coev}_c$. In unitary fusion categories, the two notions coincide.

More generally, for any morphism $f : x \to x$, where x is any object in $\mathcal{C}$, we can take its *trace* via:



where the two expressions above are equal as long as we consider the evaluation and co-evaluation maps are defined to be normalized as we did above. It is fairly straightforward to show that this definition indeed satsifies the usual cyclic property of the trace in linear algebra.

Before we get back into physics, there are two more important things to note. First off, for all simple objects $a, b, c \in \mathrm{Irr}(\mathcal{C})$, we can choose an orthonormal basis $\mathcal{B}_c^{ab}$ for the N_{ab}^c dimensional Hilbert space $\mathrm{Hom}(c, a \otimes b)$, using the inner product in Eq. 4.18. Then, we have the following relation, which we refer to as the **fusion relation**:



(4.20)

We can see this from the algebraic point of view, as for every $c' \in \mathrm{Irr}(\mathcal{C})$ if we take any $\varphi \in \mathcal{B}_c^{ab}$ and compose $\varphi^\dagger$ with the RHS of 4.20, then we have:

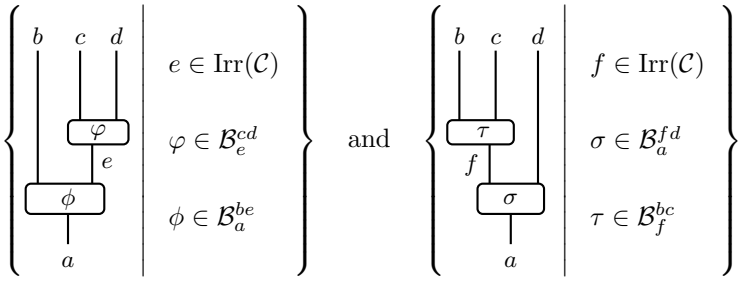
$$\sum_{c\in\mathrm{Irr}(\mathcal{C})} \sum_{\phi\in\mathcal{B}_c^{ab}} (\varphi^\dagger \circ \phi) \circ \phi^\dagger \delta_{c\,=\,c'} = \sum_{\phi\in\mathcal{B}_{c'}^{ab}} \langle \varphi \,|\phi\rangle \circ \phi^\dagger$$

$$= \sum_{\phi\in\mathcal{B}_{c'}^{ab}} \phi^\dagger \delta_{\varphi=\phi} = \varphi^\dagger \tag{4.21}$$

Since it acts as the identity on every basis element of each of the Hom-space for every simple object, the RHS of Eq. 4.20 is equal to $\mathrm{id}_{a\otimes b}$ as desired.

The last important thing we want to define is perhaps the most important underlying piece of data of any modern physical theory, the $6j$ symbols (or more generally, the F-symbol). Independently discovered by Wigner in 1940 [28] and Racah in 1942 [29] to describe associativity of SU(2) representations in the quantum theory of angular momenta, $6j$ symbols have since found uses in theories of quantum gravity, rational conformal field theories, and topological order.

Observe that for any simple objects $a, b, c, d \in \mathrm{Irr}(\mathcal{C})$, there are two distinct orthonormal bases for $\mathrm{Hom}(a \to b \otimes c \otimes d)$ given by:

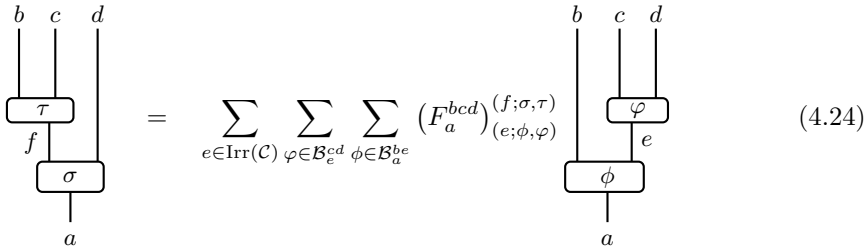


(4.22)

Thus, for each $a, b, c, d \in \mathrm{Irr}(\mathcal{C})$, there must exist a unitary operator that maps between these two bases:

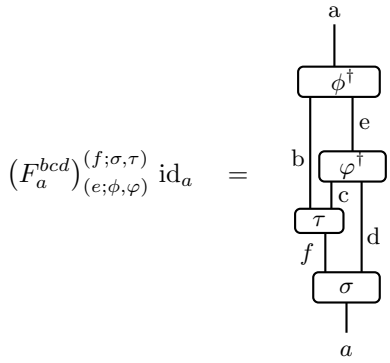
$$F_a^{bcd} : \bigoplus_{e\in\mathrm{Irr}(\mathcal{C})} \mathrm{Hom}(a, b\otimes e)\otimes\mathrm{Hom}(e, c\otimes d) \longrightarrow \bigoplus_{v\in\mathrm{Irr}(\mathcal{C})} \mathrm{Hom}(a, f\otimes d)\otimes\mathrm{Hom}(f, b\otimes c) \tag{4.23}$$

This unitary operator maps one basis vector to another via the following expression in terms of matrix elements:



(4.24)

Taking the inner product with a certain orthonormal basis vector on the right hand side gives us the following complicated, but beautiful expression for the F-symbol:



(4.25)

With this, we have built from the ground up, a complete general mathematical structure to describe topological defect lines in 2 dimensional physical systems. There is currently a lot of work being done to define mathematical structures that can describe defects in higher dimensions [30], [4].

The idea is that if we consider a $(d+1)$-dimensional theory, then its topological defects will be described by a d-fusion-category. A d-category not only has objects and morphisms, but also morphisms between morphisms called 2-morphisms, morphisms between different 2-morphisms called 3-morphisms, and so on, up until d-morphisms. In a d-fusion-category of topological defects, the objects will be codimension 1 defects, the 1-morphisms will be codimension 2 defects which are the interfaces between two higher dimensional defects, and so on, with d-morphisms being 0 dimensional points at the interface between two lines.

Building higher dimensional mathematics and finding concrete, computable examples of these higher structures appears to be the key to advancing our understanding of non-perturbative physics.

4.5 An In-Depth Look at Topological Defects in the 2d Classical Ising Model

We briefly discussed topological defects in the 1d quantum Ising model in the last chapter. However, it is much more intuitive to work with topological defects in 2 dimensions, when

time and space are on the same footing. While one may think that we need to pass to the conformal field theory setting – in which space and time are both continuous variables – to do so, there is actually a rich notion of topological defects even in the 2d classical Ising model! In fact, it was found recently that much of the rich structure in the Ising CFT can be seen even in the classical model by considering the fusion of topological defects on the lattice [31], [32]. This section will closely follow the recent paper [33], providing additional commentary along the way to tie it in with the goal of the thesis.

We begin by considering the 2d classical Ising model on a square lattice with periodic boundary conditions in zero magnetic field. For each pair of spins σ_i, σ_j or, equivalently, each link $\langle ij \rangle$ of the lattice, there is a weight of $e^{K\sigma_i\sigma_j}$ contributed to each term of partition function. Then, the partition function is given by:

$$Z(K) = \sum_{[\sigma]} \prod_{\langle ij \rangle} e^{K\sigma_i\sigma_j} \prod_i d_i \tag{4.26}$$

where $d_i \in \mathbb{C}^*$ is a constant factor which we have the freedom to include without changing the partition function. Morally, it can be set to 1 for all i, but other values will be useful if the partition function has any defects. The notation here is intentionally suggestive of the quantum dimension from the previous section.

We know that in the 1d quantum Ising model, the global $\mathbb{Z}_2$ symmetry can be reconstructed from local defects on links. Recall that if a defect was present on a link, then the local term of the Hamiltonian supported on the link was modified as $\sigma_i^z\sigma_{i+1}^z \mapsto -\sigma_i^z\sigma_{i+1}^z$. Since the rows configurations of the 2d classical Ising model in the transfer matrix picture are formally equivalent to that of a 1d quantum Ising model, we may expect that this defect will also arise in the classical case.

Indeed, if we fix a path γ on the dual lattice, there is a topological defect $\mathcal{D}_\epsilon(\gamma)$ which acts by changing the weight of each link of the original lattice $\langle ij \rangle$ that it intersects via $e^{K\sigma_i\sigma_j} \mapsto e^{-K\sigma_i\sigma_j}$. We write the denote the resulting modified partition function by $Z_{\mathcal{D}_\epsilon(\gamma)}$. $\mathcal{D}_\epsilon(\gamma)$ is topological as can be deformed locally as in Figure 4.7 at any point without changing the partition function:

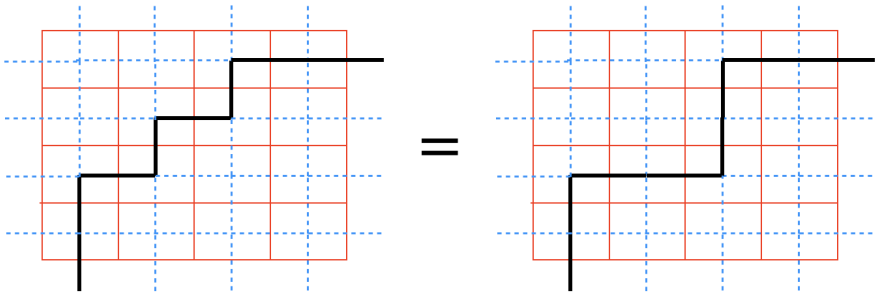


Figure 4.7: Local deformation of $\mathcal{D}_\epsilon(\gamma)$ to $\mathcal{D}_\epsilon(\gamma')$

This follows from the fact that if the sites where the deformation happens are labeled as in

Figure 4.8, then we can pull out the factors of the partition which have σ_0 and then sum over just σ_0 portion of $[\sigma]$ to see that:

$$\sum_{\sigma_0=\pm 1} e^{-K\sigma_1\sigma_0}e^{-K\sigma_2\sigma_0}e^{K\sigma_3\sigma_0}e^{K\sigma_4\sigma_0} = \sum_{\sigma_0=\mp 1} e^{K\sigma_1\sigma_0}e^{K\sigma_2\sigma_0}e^{-K\sigma_3\sigma_0}e^{-K\sigma_4\sigma_0} \tag{4.27}$$

where all we did was rename $\sigma_0 \mapsto -\sigma_0$. This immediately implies that the deformation in Figure 4.7 holds, i.e:

$$Z_{\mathcal{D}_\epsilon(\gamma)} = Z_{\mathcal{D}_\epsilon(\gamma')} \tag{4.28}$$

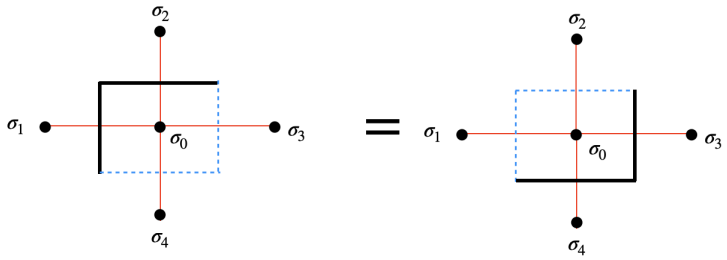


Figure 4.8: Close-up of Figure 4.7 at the site of deformation

From the exact same trick of renaming the spin of a single site σ_0, we can also deform $\mathcal{D}_\epsilon(\gamma)$ locally as:

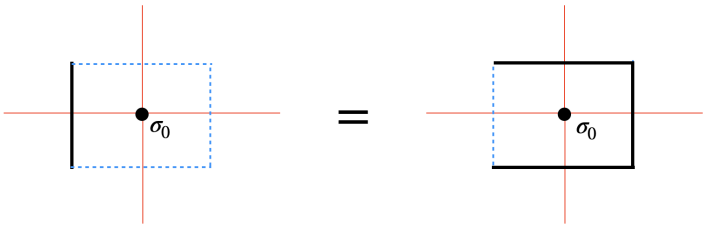


Figure 4.9: Other kind of local deformation of $\mathcal{D}_\epsilon(\gamma)$

More interestingly, we also have the relations in Fig. 4.10, which we refer to as the bubble-popping and the recoupling relations:

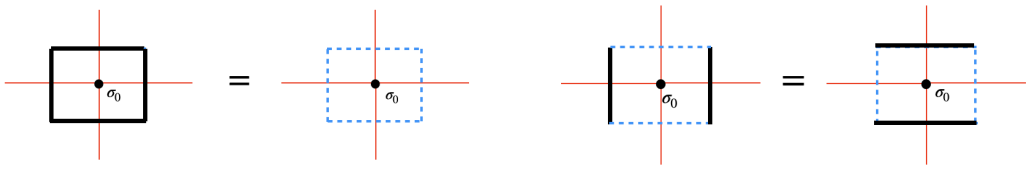


Figure 4.10: Bubble popping (left) and recoupling (right) relations for $\mathcal{D}_\epsilon$

It's important to note that any expectation values or spin correlation functions are also

invariant under the deformation of topological defects, so long as there are none of the inserted spins are in the way of the deformation. For instance, consider the correlation function:

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{Z(K)} \sum_{[\sigma]} \sigma_i \sigma_j \prod_{\langle kl \rangle} e^{K \sigma_k \sigma_l} \prod_i d_i \tag{4.29}$$

Clearly, none of the relations in Figures 4.8, 4.9, or 4.10 will hold on sites i or j, as simply renaming σ_i or σ_j would change the relative sign of the terms in the partition function. We visualize these *operator insertions* as punctures on sites which we cannot *freely* deform defect lines across.

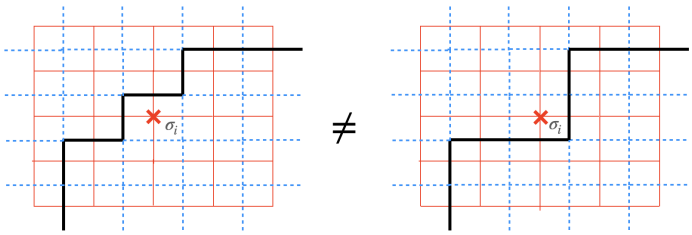


Figure 4.11: Operator insertion of σ_i

From Eq. 4.29 it is clear, however, that we can freely deform the defect at any vertices that do not have operators inserted without changing the partition function.

Now, observe that bubble popping and recoupling relations imply that $\mathcal{D}_\epsilon$ is locally invertible, as can be seen in Figure 4.12. That is, within a small patch of the lattice with no operator insertions and no global topological obstructions, the presence of two adjacent parallel $\mathcal{D}_\epsilon$ lines has cancel to the *trivial defect*. This reflects the fusion rule of the corresponding topological line defects: applying the spin-flip symmetry twice returns the system to its original configuration. Thinking in terms of fusion categories, we say that:

$$\mathcal{D}_\epsilon \otimes \mathcal{D}_\epsilon \simeq \mathbb{1} \tag{4.30}$$

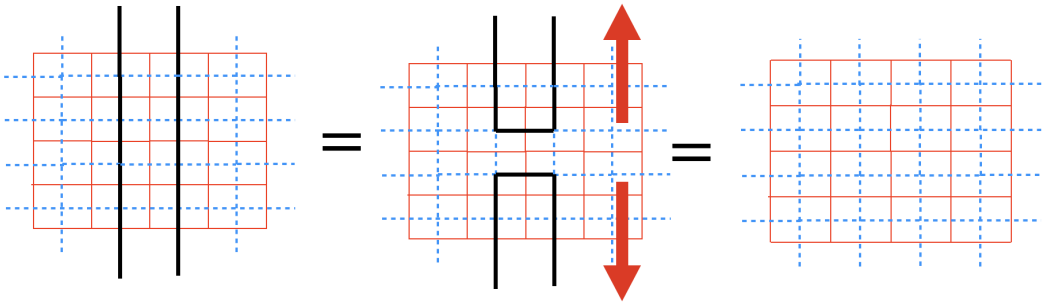


Figure 4.12: Local cancellation of two parallel $\mathcal{D}_\epsilon$ defects

Remark.

> It is important to note that there is some subtlety here. If the $\mathcal{D}_\epsilon$ lines are finite in extent —that is, they are do not form closed loops around the torus or do not stretch from boundary to boundary —then this cancellation is incomplete. One can locally apply the reconnection move and deform the lines outward, but there will always remain two endpoints where the defect lines began and ended. These endpoints are not topologically trivial: they may carry nontrivial data, or require operator insertions to fully absorb the defect.

Similarly to what we did in the 1d quantum case, we can also gauge the global $\mathbb{Z}_2$ symmetry in the 2d classical case. In fact, this leads to another topological defect that is associated with Kramers-Wannier duality!

We begin by defining the non-simple defect $A = \mathbb{1} \oplus \mathcal{D}_\epsilon$, where $\mathbb{1}$ refers to the trivial defect. To do so, we choose a path γ on the dual lattice and assign a $\mathbb{Z}_2$ variable $s_l \in \{\pm 1\}$ to each edge l of the path. A then modifies the weight of each link that it crosses via:

$$e^{K\sigma_i\sigma_j} \mapsto e^{s_l K\sigma_i\sigma_j} \tag{4.31}$$

effectively acting as $\mathbb{1}$ if $s_l = +1$ and as $\mathcal{D}_\epsilon$ if $s_l = -1$. As A is a direct sum, we want the $\mathbb{1}$ and $\mathcal{D}_\epsilon$ components to be somewhat disjoint; it does not make sense to have a path that alternates between acting trivially and non-trivially on every other site, for instance. Thus, we enforce that at at each point, the two adjacent edge labels s_l and $s_{l'}$ have to agree. Thus, the path is *either* entirely $\mathcal{D}_\epsilon$ or entirely $\mathbb{1}$.

We then implement A by summing over all configurations $[s]$ of the edge variables along the path subject to the constraint above, which gives rise to the direct sum.

Now, we also want to define junctions between two paths; these will play the role of the Gauss Laws in the 1d case. To a trivalent junction at a vertex v on the dual lattice with edge labels as in Figure 4.13 below, we assign a weight of:

$$\frac{1}{\sqrt{2}}\delta(s_1 s_2 s_3, 1) \tag{4.32}$$

which serves to ensure that three A defects can only meet at a vertex if they are consistent with the fusion rules, just as we did with the Gauss law in the previous section.

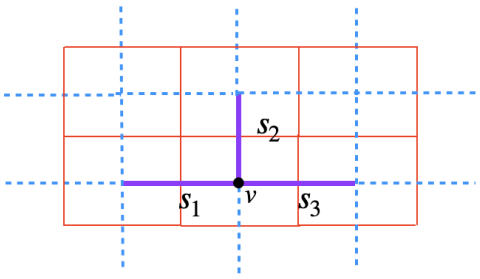


Figure 4.13: Trivalent junction of A defects (purple lines) at a vertex v

This implies the following local relation for the A defects in Figure 4.14. In equation form, this is simply saying that summing over s_2 and s_3 in the left-most diagram with combined weight:

$$\frac{1}{2}\,\delta(s_1 s_2 s_3, 1)\delta(s_2 s_3 s_4, 1) \tag{4.33}$$

gives two terms which enforce the relation $s_1 = s_4$.

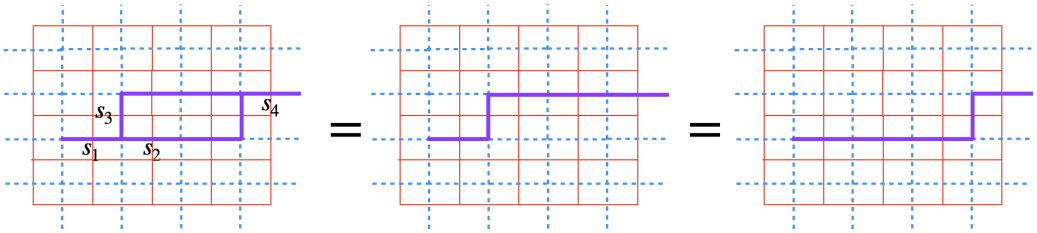


Figure 4.14: Canceling loops of A defects

With this identity, we can consistently define a four-valent vertex in terms of 2 trivalent vertices, as in Figure 4.15:

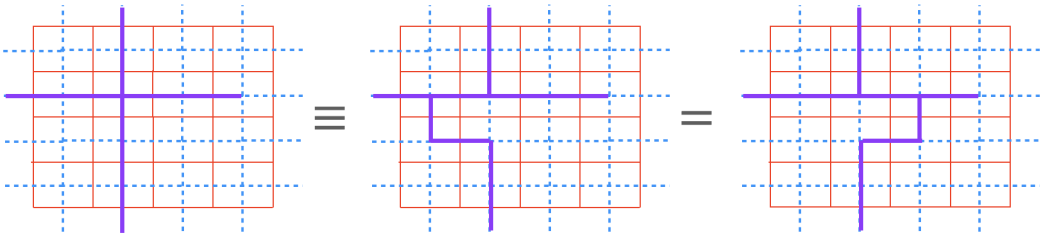


Figure 4.15: Definition of four-valent vertex of A defects

From this, it is straight forward to find that the weight of the four-valent vertex with edge labels $s_1, \cdots, s_4$ is given by:

$$\frac{1}{2}\,\delta(s_1 s_2 s_3 s_4, 1) \tag{4.34}$$

This just says that there should be an even number of -1 edge labels surrounding each vertex. This is the exact same condition we found for the k_l variables on the links way back in our original derivation of Kramers-Wannier duality! Thus, we know that this condition means that configurations with closed loops of $\mathcal{D}_\epsilon$ defects are the only terms which will contribute to the partition function

We can now finally implement gauging through defects. Consider a network of A defects inserted on every link of the dual lattice below a certain horizontal line G on the dual lattice:

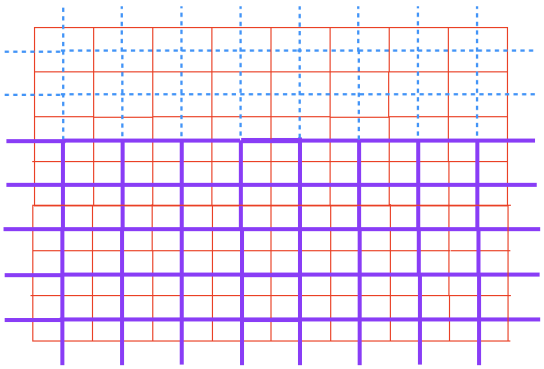


Figure 4.16: Gauging via network of A defects

This modifies the partition function as:

$$\sum_{[s]}\sum_{[\sigma]}\prod_{v}\frac{1}{2}\delta(s_{e_1}s_{e_2}s_{e_3}s_{e_4},1)\prod_{\langle ij\rangle \text{ above } G}e^{s_{\langle ij\rangle}\sigma_i\sigma_j}\prod_{\langle kl\rangle \text{ above } G}e^{\sigma_k\sigma_l}\prod_i d_i \tag{4.35}$$

where the product over v denotes vertices v of the dual lattice surrounded by edges $e_1, \cdots, e_4$.

Recall the identification of the vertical axis of the 2d classical Ising model with the imaginary time in the 1d quantum model. Thus, placing a network of A defects below a certain horizontal line is simply a graphical representation of applying the gauging map to the Hilbert space of the usual 1d quantum Ising model at a given moment in time.

The interface between $\mathcal{H}$ and $\mathcal{H}_{\text{gauged}}$ is in fact itself a topological defect! Indeed, observe that we can deform the boundary between the network of A defects and the trivial defects however we please while leaving the partition function in Eq. 4.35 invariant. Specifically, we have the following local relations in Figure 4.17:

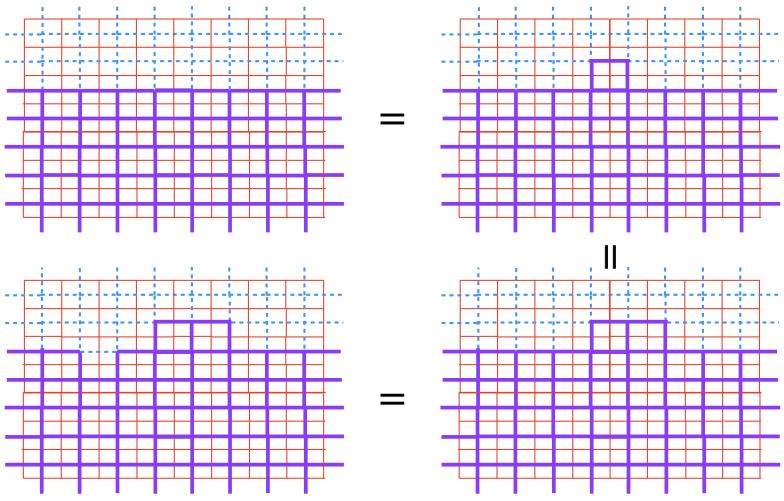


Figure 4.17: Local deformations of the gauging interface

For instance, the first equality holds by performing the sum over the added defect, deforming the $\mathcal{D}_\epsilon$ term in the sum back onto the original interface, and then simply relabeling the link variable that it retracts to from $s_l \mapsto -s_l$ in the sum. The other equalities follow from the same sorts of tricks of summing a link variable, deforming one of the terms, and relabeling.

We denote this topological boundary line between the network of A defects and the original lattice by G, and refer to it as the *gauging defect line*. The fact that even the gauging map has a topological interpretation speaks to the ubiquity of topological defects in physics. Gauging is a very general procedure that applies to countless physical models, including the standard model of particle physics. Thus, there is potentially a lot to gain from trying to implement gauging as a topological interface in other lattice models and QFTs with global symmetries. Similarly, that the gauging map has an associated topological defect implies that even more exotic symmetries like Kramers-Wannier duality actually arise from topological foundations just like traditional global symmetries like the $\mathbb{Z}_2$ symmetry.

Remark.

We will not show it here as we have none two very similar computations to derive Kramers-Wannier duality in other ways, but it can be shown that there is an invertible topological defect D supported diagonally between spins and dual spins (Figure 4.18) that can be fused with G to swap the roles lattice and dual lattice and map the partition function as:

$$Z_{\text{gauged}}(K) \mapsto \tilde{Z}(\tilde{K}) \tag{4.36}$$

where $\tilde{Z}(\tilde{K})$ denotes an Ising model at the dual coupling $\tilde{K}$ whose spins lie on the dual lattice [33].

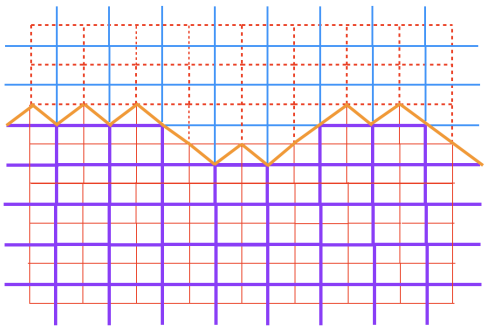


Figure 4.18: Invertible D defect (orange line) mapping original gauged Ising model to dual Ising model

This invertible defect D is simply a topological defect representation of the unitary operator U made up of Hadamards and controlled Z operators which we used in the previous chapter to change basis from the gauged Hamiltonian to the dual Hamiltonian. The composition of G and D is then the non-invertible KW map that implements Kramers-Wannier duality.

The gauging defect line G has its own fusion rules which we can derive directly on the

lattice. For instance, we can see that a gauging defect G with a D_ψ defect running parallel leaves G invariant:

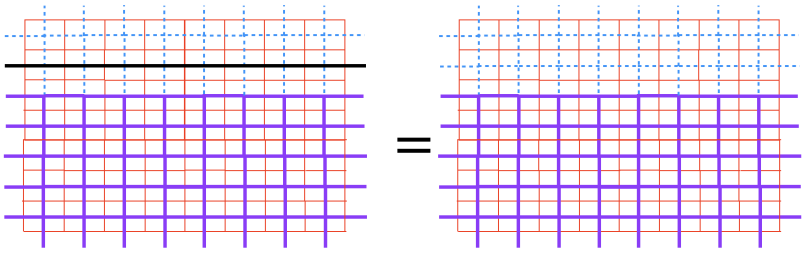


Figure 4.19: Fusion of G and D

This follows from deforming the D defect downward into the gauging network using 4.9 and then relabeling $s_l \mapsto -s_l$ for every link variable on the boundary of the interface (since the deformation switches their sign). This can be summarized by the fusion rule:

$$G \otimes \mathcal{D}_\epsilon \simeq \mathcal{D}_\epsilon \otimes G \simeq \mathbb{1} \tag{4.37}$$

We can also fuse two parallel gauging defects. Consider a gauging defect G and another G^* whose interface is on the adjacent site to G's but whose gauge network extends in the opposite direction, as in Figure 4.20:

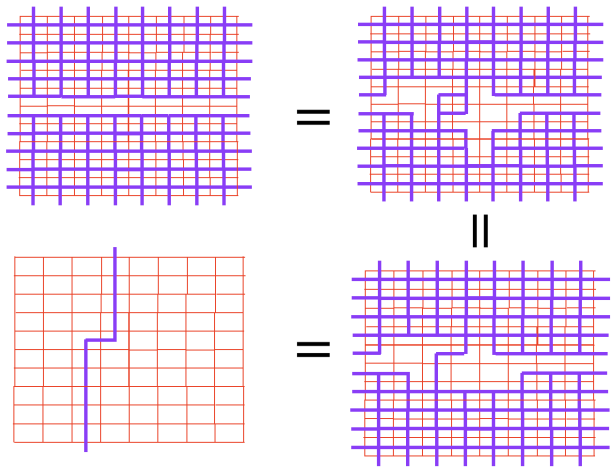


Figure 4.20: Two adjacent gauge interfaces fusing together

As we can see, by deforming the boundaries of the two interfaces, connecting them, using the loop cancellation rule from Figure 4.14, and then pulling back the remaining gauge interfaces, we find that we have the **local** fusion rule:

$$G \otimes G \simeq A = \mathbb{1} \oplus \mathcal{D}_\epsilon \tag{4.38}$$

Thus, the gauge defect G is non-invertible, as there is no object that it can fuse with to yield the trivial defect on its own. The best we can do is fuse it with another gauge defect

G^* to obtain a direct sum of the trivial defect and the $\mathbb{Z}_2$ defect. This exactly matches our findings regarding the gauging map in the 1d quantum case.

To summarize, starting from the symmetries of the 2d classical Ising model we were able to find three simple topological defects: the $\mathbb{Z}_2$ defect $\mathcal{D}_\epsilon$, the gauging defect G, and the trivial defect $\mathbb{1}$. These three defects obey the following non-trivial fusion rules:

$$\mathcal{D}_\epsilon \otimes \mathcal{D}_\epsilon \simeq \mathbb{1} \qquad \mathcal{D}_\epsilon \otimes G \simeq G \otimes \mathcal{D}_\epsilon \simeq \mathbb{1} \qquad G \otimes G \simeq \mathbb{1} \oplus \mathcal{D}_\epsilon \tag{4.39}$$

These a fusion rules generate a fusion category **Ising**, whose F-symbols and quantum dimensions can actually be found directly from local interactions on the lattice!

If you recall, these are the exact same fusion relations of the primary fields or Verlinde lines the Ising CFT! In fact, it was found in that by viewing the spin-flip and gauge/duality defects in terms of Dehn twists of the torus and manipulating these defects using the F symbols, we can exactly derive the conformal weights $\frac{1}{2}$ and $\frac{1}{16}$ of the corresponding primary operators in the Ising CFT [31]. The fact that we can derive much of the rich algebraic structure of the Ising CFT directly from the associated classical Ising model —not even at criticality —suggests that much of the rich algebraic structure of conformal field theory may not be inherent results of its conformal symmetry. The fundamental aspects of solvable QFTs appear to be more dependent on topological defects and the categorical data that governs their interactions.

Chapter 5

Conclusion

In this thesis, we explored how rich mathematical structures emerge from studying one of the simplest models in statistical mechanics: the two-dimensional Ising model. Beginning with an information-theoretic formulation of classical statistical mechanics, we examined how the Ising model encodes symmetries and dualities that extend naturally into the quantum and continuum settings.

Through the transfer matrix formalism, we connected the 2D classical Ising model to the 1D quantum transverse field Ising chain, and from there, followed a path through duality transformations and fermionization techniques to arrive at the free Majorana fermion conformal field theory. Along the way, we saw how global symmetries of a Hamiltonian can be completely reconstructed from local topological defects. We also saw how summing over the insertions of topological defects on the links of the lattice and imposing a Gauss law, a process known as gauging, is the most natural way of thinking about Kramers-Wannier duality.

This discussion motivated the introduction of category theory as a framework for understanding symmetries and defects in a more general and abstract way. Starting from ordinary categories, we defined properties and structures like direct sums, idempotent completeness, and semisimplicity to lead us to the notion of 2-vector spaces —a higher dimensional notion of usual vector spaces better equipped to deal with the problems of the modern theoretical physicist. On top of this, we also defined a monoidal product which came with a graphical calculus that gave credence to our hypothesis about category theory being the natural language of topological defects. We then explained the notion of rigidity in a monoidal category and formally defined unitary fusion categories. After studying some properties of unitary fusion categories, we jumped right back into the 2d classical Ising model with our newfound knowledge. We saw how there exists three simple defects in the 2d Ising model —the trivial defect, the $\mathbb{Z}_2$ defect, and the gauging, or KW, defect —which fuse together when on adjacent links according to the fusion rules of the Ising fusion category.

While many of the results presented are known, the goal of this thesis was to bridge a gap that I perceived to be missing in the literature. Namely, a self-contained and physically motivated introduction to topological defects and categorical symmetries. My hope is that

this thesis serves as a coherent narrative which builds up toward the categorical language that underlies modern approaches to quantum field theory and lattice models.

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