Notes on Unitary TQFTs and Stable Homotopy

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(*Based on a course given by Luuk Stehouwer*)

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Abstract

Despite being one of the central pillars of any quantum theory of physics, in-depth treatments of unitarity tend to be ignored in functorial descriptions of even non-extended TQFTs. In these lectures, we will make precise the notion of a unitary TQFT while exploring invertible TQFTs and their interactions with stable homotopy theory along the way.

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1 Lecture 1: TQFTs and Rudiments of Algebraic Topology

The main goal of these lectures will be to carefully define non-extended **unitary** TQFTs. One side quest we will explore along the way will be to restrict to a very specialized case of TQFTs, so-called invertible TQFTs, which we can use the powerful framework of stable homotopy theory and spectra to classify and understand. Another side quest will be to give some remarks on the *extension* of our unitary TQFT framework to the world of (∞, n)-categories.

Given that the first word of **T**QFT is *topological*, and since we are assuming very little background knowledge of topology, much of this lecture will be devoted to teaching topology. Specifically, if we want a QFT with fermions and unitarity (like in the real world), there are some subtle concepts we will need.

Let us begin by briefly discussing the last three letters of T**QFT**. There are countless ways to formalize the most fundamental aspects of quantum field theory; in these lectures, we will focus on the good old-fashioned Schrodinger picture of quantum theory, which is formalized by the Atiyah-Segal axioms.

The idea of this picture is the following core principle underlying essentially all of physics: given knowledge of a system in the present, how do we predict its future? This motivates the following heuristic definition:

Definition 1.1: Quantum Field Theory*

An **n-dimensional QFT** "should" be a rule which

1. assigns to any closed $(n-1)$ dimensional manifold Y^{n-1} a Hilbert space $Z(Y)$, which is the space of states at a given time slice

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2. and for any n-dimensional spacetime M connecting two spaces Y_1 and Y_2, a linear map $Z(M): Z(Y_1) \to Z(Y_2)$, which we call the **time evolution operator.**

3. It should additionally satisfy $Z(Y_1 \sqcup Y_2) = Z(Y_1) \otimes Z(Y_2)$ for any spaces Y_1 and Y_2.

The third bullet point axiomatizes some aspects of the fundamental notion that physics is **local.** It essentially says that two systems that are space-like spearated should not affect each other.

We will formalize this idea in the language of category theory for a special, but fairly general case, by the end of this lecture. It's important to note that the above 'definition' does not capture the full *extent* of locality, which in principle says that we can build up our theory by gluing together pieces of arbitrary codimension (this involves higher categories).

Let us now move into the world of topology to begin to make our idea rigorous. We start with the following definition

Definition 1.2: Bordism

A **bordism**[a] from a closed manifold Y_1^{n-1} to another closed manifold Y_2^{n-1} is a compact manifold M^n equipped with the following data:

1. A partition $\partial M = \partial_{\text{in}} M \sqcup \partial_{\text{out}} M$
2. Diffeomorphisms $Y_1 \simeq \partial_{\text{in}} M$ and $Y_2 \simeq \partial_{\text{out}} M$.[b]

[a]Sometimes people will refer to this as a cobordism. The 'co-' prefix is in fact meaningless here.

[b]In principle, we should include as part of the data embeddings of infinitesimal collar neighborhoods $\theta_{\text{in}} : Y_1 \times [0, \epsilon) \to M$ and $\theta_{\text{out}} : Y_2 \times (-\epsilon, 0] \to M$ which restrict to diffeomorphisms at the boundaries. This gives us a natural ingoing and outgoing state and an explicit means of gluing bordisms together

Having a distinguished "arrow of time", at least locally near the boundaries, is crucial for defining further geometric structures on bordisms. One way to see this is the following. Given a bordism M, we get a short exact sequence of bundles

$$0 \to T(\partial M) \to TM|_{\partial M} \to \text{normal}(\partial M \hookrightarrow M) \to 0. \tag{1}$$

Choosing an outward "normal" vector trivializes the normal line bundle as $TM|_{\partial M} \simeq T\partial M \oplus \underline{\mathbb{R}}$[1]. This decomposition is exactly what is needed for us to say that a geometric structure on M induces a corresponding structure on ∂M.

There is a caveat here insofar as the notion of a normal bundle is only meaningful if M is equipped with a metric. However, the space of metrics for a given manifold is convex and hence a choice of a metric is "contractible data;" since we will be working homotopy theoretically, a choice of a metric will not matter to us. Similarly, up to homotopy, there are two possible choices of trivialization of the normal bundle, but these two choices simply differ by $-1 \cdot \text{id}_{\mathbb{R}}$.

So far, we have been working with smooth information; we would like to convert this into some kind of homotopical information. First "recall" the following concepts from topology.

Remark 1.1: Classifying Spaces of Real Vector Bundles

Rank r real vector bundles $V \to X$ are "classified" by homotopy classes of maps $f : X \to BO(r) =$

[1]$\underline{\mathbb{R}}$ denotes the trivial $\mathbb{R}$ line bundle over M.

$\mathrm{Gr}\left(\mathbb{R}^r \subset \mathbb{R}^\infty\right)^a$ in the sense that V is given by the pullback bundle of f:

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[d, dashed] & EO(r) \arrow[d] \\ X \arrow[r, "f"'] & BO(r) \end{tikzcd} . \tag{2}$$

Here, $EO(r)$ is the canonical $\mathbb{R}^r$ bundle over $BO(r)$, where the fiber over a given subspace is the collection of all possible choices of its orthonormal bases. This space is contractible with a natural free $O(r)$ action.

Formally, $BO(r)$ (and classifying spaces BG of a general group G) is constructed by viewing the ∞-group $O(r)$ as an ∞-groupoid with a single object and "recalling" that ∞-groupoids are just topological spaces. Morally, we should think of this map f as just continuously assigning a copy of $\mathbb{R}^r$ to every point of the base space X.

[a] $\mathrm{Gr}(\mathbb{R}^r \subset \mathbb{R}^\infty)$ denotes the Grassmanian manifold of r-dimensional subspaces of infinite dimensional real space

It is imporant to note that an isomorphism of vector bundles

$$\begin{tikzcd} V_1 \arrow[r, "\sim"] \arrow[d] & V_2 \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd} \tag{3}$$

gives rise to a a homotopy between maps into $BO(r)$

$$\begin{tikzcd} X_1 \arrow[d] \arrow[dr] & \\ X_2 \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow"] & BO(r) \end{tikzcd} \tag{4}$$

This is to say, in modern categorical language, that we can study vector bundles by looking at the *slice category* over $BO(r)$ in some category of topological spaces.

As an example, if two manifolds M_1 and M_2 have isomorphic tangent bundles, the homotopy between their classifying maps is precisely given by the familiar pushforward

$$\begin{tikzcd} X_1 \arrow[d, "f"'] \arrow[dr] & \\ X_2 \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow df"] & BO(r) \end{tikzcd} . \tag{5}$$

With this, we can finally define the following

Definition 1.3: Tangential Structure

An $(X \xrightarrow{\rho} BO(r))$ tangential structure on M is a map $M \to X$ and a homotopy

$$\begin{tikzcd} & X \arrow[d, "\rho"] \\ M \arrow[ur] \arrow[r, "TM"'] \arrow[ur, phantom, shift right=3, "\Leftarrow"] & BO(n) \end{tikzcd} \tag{6}$$

The idea is that by factoring the tangent bundle through some other space (up to homotopy), we can introduce interesting structures and symmetries on the tangent bundle.

It is an interesting remark that morphisms of tangential structures will always involve tetrahedra. Above we can see that the tangential structure itself is a triangular diagram with a two-cell filled in by the homotopy; furthermore, we can imagine that when defining morphisms between these and even higher morphisms between those that we will do so by filling in higher and higher simplices.

Before we get into examples, we must refine our structure a bit more. Ultimately we want to define tangential structures on bordisms; this will require us to have a stable and consistent way of extending and restricting tangential structures to manifolds of different dimensions.

Notice that since every point in $BO(n)$ is some n-dimensional linear subspace, there exist a sequence of embeddings

$$\cdots \xrightarrow{-\oplus\mathbb{R}} BO(n) \xrightarrow{-\oplus\mathbb{R}} BO(n+1) \xrightarrow{-\oplus\mathbb{R}} \cdots \tag{7}$$

given by simply taking a point-wise direct sum with $\mathbb{R}$. Pulling back maps $X \to BO(r) \to BO(r+1)$ as above classifies bundles of the form $E \oplus \underline{\mathbb{R}}$ over a space X where E is an arbitrary rank r bundle over X.

We can then define the stable classifying space of real vector bundles as

$$BO := \operatorname{colim}_{n\to\infty}\left(\cdots \xrightarrow{-\oplus\mathbb{R}} BO(n) \xrightarrow{-\oplus\mathbb{R}} BO(n+1) \xrightarrow{-\oplus\mathbb{R}} \cdots\right). \tag{8}$$

Intuitively, this space is identifying n-dimensional bundles classified by $BO(n)$ with those that sit inside of $BO(n+1)$ that only differ by a copy of a trivial bundle. Thus, the stable classifying space BO has no real way to distinguish vector bundles of different rank, it just keeps track of the stable geometry by classifying them up to the addition of trivial bundles.

It is also important to note that by taking direct sums of bundles, we obtain maps

$$BO(n) \times BO(m) \xrightarrow{\oplus} BO(n+m), \tag{9}$$

which upon passing to the stable space, provides a multiplication

$$BO \times BO \to BO. \tag{10}$$

We won't discuss it further, but in some sense we can formally add inverses $\ominus$ via a pointwise "orthogonal complement". Since the product is commutative up to isomorphism ($V \oplus W$ is not literally equal to $W \oplus V$ for vector spaces), we say that $(BO, \oplus)$ forms an $\mathbb{E}_\infty$ group! This is deeply related to topological KO theory, but I digress.

Definition 1.4: Stable Tangential Structure

A **stable** tangential structure is a map $X \to BO$. This gives a sequence of $X(n)$ structures determined by the homotopy pullback :

$$\begin{tikzcd} X(n) \arrow[r, dashed] \arrow[d, dashed] & BO(n) \arrow[d] \\ X \arrow[r] & BO \end{tikzcd} \quad . \tag{11}$$

Homotopy pullbacks are sometimes also called homotopy fiber products and hence the above pullback can be denoted as $X \times_{BO} BO(n)$.

You can also have tangent structures on bundles themselves. In fact, from the universal property of the homotopy pullback it follows that an $X(n+k)$ structure on $TM \oplus \underline{\mathbb{R}^k}$ is equivalent to giving an $X(n)$ structure on TM. Intuitively this is because additions of the trivial bundle are stabilized in BO such that when we pull back, these differences are not visible.

Example 1.5: Stable Tangential Structures

Now, we can discuss some examples of stable tangential structures.

1. $X = BO$ with the identity map is a trivial tangential structure. There is no additional data given by this choice.

2. Define BSO as $\pi_{\geq 2}BO$, meaning that we formally kill off the nontrivial $\pi_1 BO = \mathbb{Z}_2$; this amounts to defining BSO to be the universal covering space of BO.

   If we take $X = BSO$ with the corresponding double covering map over BO, we get an **orientation** over a space M. Specifically, this gives us the following diagram

$$\begin{tikzcd} & BSO \arrow[d] & \\ X \arrow[r] \arrow[ur, dashed] & BO \arrow[r, "\det"'] & B\mathbb{Z}_2 \end{tikzcd} \tag{12}$$

   This map $\det : BO \to B\mathbb{Z}_2$ is precisely the first Stiefel-Whitney class w_1; it can alternatively be defined as a map from $\pi_1 \to \mathbb{Z}_2$, which measures whether parallel transport around loops preserves orientation. This exactly measures the obstruction to defining an orientation on a manifold.

3. Define $\mathrm{BSpin} := \pi_{\geq 3}BO$, killing off the fundamental group as well as the second homotopy group $\pi_2 BO = \mathbb{Z}_2$. This defines a **spin structure** on M, so long as we have that the second Stiefel-Whitney class

$$M \to BO \xrightarrow{w_2} B^2\mathbb{Z}_2 \tag{13}$$

   vanishes.

   It is very important to note that we can not naively apply this definition of BSpin rank-wise, as $\mathrm{BSpin}(n) \neq \pi_{\geq 3}BO(n)$ if $n \geq 3$!

4. Choosing $X = \mathrm{pt}$ literally amounts to choosing a **framing** of M, i.e a consistent choice of basis of T_xM for all $x \in M$.

5. Lastly, we can consider taking $X = BO \times BG \xrightarrow{p_1} BO$, where p_1 is the projection onto the first factor and $BG = K(G, 1)$ is the classifying space of a finite group G (constructed by expressing G as an ∞-groupoid with a single object).

   Since the map to the BO factor is already fixed by the tangent bundle, a $X = BO \times BG$ structure is specified exactly by a map from $M \to BG$, which is simply a **principal G-bundle over M** (also referred to as a G-local system).

With this, we can finally add additional structure to bordisms.

Definition 1.6: X-Structured Bordism

The **X-structured bordism category** $\mathrm{Bord}^X_{n,n-1}$ has **objects** Y^{n-1} with $X(n-1)$-structure and **morphisms** $Y_1 \to Y_2$ given by bordisms M with $X(n)$ structure along with the following **additional data**:

1. An isomorphism $TY_2 \oplus \underline{\mathbb{R}} \simeq T\partial_{\mathrm{out}}M \oplus \underline{\mathbb{R}} \simeq TM|_{\partial_{\mathrm{out}}M}$ that preseres the $X(n)$-structure

2. Another isomorphism $TY_1 \oplus \underline{\mathbb{R}} \simeq T\partial_{\mathrm{in}}M \oplus \underline{\mathbb{R}} \overset{\mathrm{id}\oplus -\mathrm{id}_{\mathbb{R}}}{\simeq} T\partial_{\mathrm{in}}M \oplus \underline{\mathbb{R}} \simeq TM|_{\partial_{\mathrm{in}}M}$

up to $X(n)$-preserving diffeomorphisms relative to the boundary. We need to quotient out by these diffeomorphisms so that the gluing of bordisms (composition of morphims) makes $\mathrm{Bord}^X_{n,n-1}$

a strict category, rather than an $(\infty, 1)$ category. The identity morphism is then simply given by cylinders $Y \times [0, 1]$.

While the difference in data for the ingoing and outgoing boundaries is subtle, it is simply described by the boundary normal vectors in the bordism in Fig. 1.

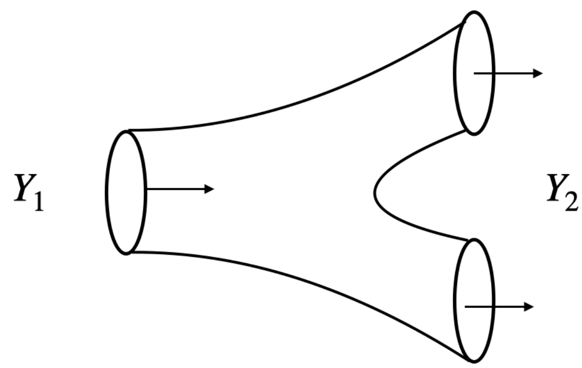


Figure 1: Tangential Stucture Preserving Bordism

Now, we are almost ready to define a solid definition of TQFT which will be able to support fundamental physical concepts like unitarity and fermions.

"Recall" that an $\mathbb{E}_\infty$ category is an $\mathbb{E}_\infty$-monoid internal to Cat. This amounts to a category $\mathcal{C} \in$ Cat with a functor $- \otimes - : \mathcal{C} \times C \to C$ with homotopies (natural transformations) filling all possible braiding, associating, and unitor diagrams. In particular this means, that we have natural transformations:

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "\text{flip}"'] \arrow[dr, phantom, "\Downarrow\beta"] & \mathcal{C} \\ \mathcal{C} \times \mathcal{C} \arrow[ur, "\otimes"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes \times id"] \arrow[d, "id \times \otimes"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "\alpha"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C} \end{tikzcd} . \tag{14}$$

The pentagon and hexagon relations and even the diagram relating $\beta_{x,y} \circ \beta_{y,x} \simeq id_{x \otimes y}$ must all be filled by 3-cells. But since there are no available 3-morphisms in the 2-category Cat, all of these relations must hold on-the-nose. This means that an $\mathbb{E}_\infty$-category is just a fancy, but more desciptive and generalizable, name for a **symmetric monoidal category.**

Example 1.7: $\mathbb{E}_\infty$ Categories

There are a few important examples of $\mathbb{E}_\infty$ categories needed to define TQFTs.

1. One example is the category $(\text{Bord}^X_{n,n-1}, \sqcup)$
2. Another example is $(\text{Vect}_k, \otimes)$.
3. Lastly is the category of **super vector spaces** $(\text{sVect}_k, \hat{\otimes}_k)$. A super vector space is a $\mathbb{Z}_2$ graded vector space $V = V_0 \oplus V_1$; we say that V_0 is bosonic, while V_1 is fermionic. We furthermore define the tensor product to respect this grading, i.e

$$(V \otimes W)_i = \bigoplus_{j+k=i} V_j \otimes W_k \tag{15}$$

and the braiding on homogeneous elements (vectors which lie entirely in either the bosonic or fermionic component) to be given by the **Koszul sign rule**

$$\beta_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v, \tag{16}$$

where $|\cdot|$ denotes the parity of the vector.

Finally, we can define a TQFT!

Definition 1.8: TQFT

An **n-dimensional X-structured TQFT** is an $\mathbb{E}_\infty$ functor $Z : \mathrm{Bord}^X_{n,n-1} \to \mathrm{sVect}_\mathbb{C}$.

Note that the target space of sVect lends much more naturally to physical models than the usual target space of Vect, as this sVect allows us to consider theories with fermionic degrees of freedom; we recover the usual definition of a TQFT if the image of Z lands in Vect in which case we say that the TQFT is **bosonic**. Furthermore, it follows from the definition that Z factors through the category of finite-dimensional super vector spaces.

As a final remark, it is important to mention the case when $X = BO \times BG$ for a finite group G. Recall that this X-structure will define a principal G-bundle over the bordism. Principal G-bundles over M are in 1:1 correspondence with $\mathrm{Hom}(\pi_1(M) \to G)/\text{conjugation}$.[2] Suppose a closed manifold Y^{n-1} has a trivial G-bundle over it, in the sense that it has trivial monodromy. Then, for every $g \in G$, there is an extension to the bordism $M_g = Y \times [0,1]$ which gauge transforms the bundle over the outgoing boundary by g.

Thus, when passing to the TQFT, for every $g \in G$, we get a map

$$Z(M_g) : Z(Y) \to Z(Y) \tag{17}$$

coming from the automorphisms of the background G-bundle.

Functoriality then tells us that this gives us a representation $G \to \mathrm{Aut}(Z(Y))$ of G on the state space of Y. Thus, by coupling a TQFT to background G gauge fields, i.e by supplying $X = BO \times BG$ tangential structure on the bordism category, we can describe theories with non-anomalous G-symmetries.

2 Lecture 2: Dagger Structures, Rigidity, and Unitarity

Last lecture, we said that a QFT should map $Y^{n-1} \to Z(Y) \in \mathrm{Hilb}_\mathbb{C}$; however, we contradicted ourselves by only using (s)$\mathrm{Vect}_\mathbb{C}$ as our target space, not (s)Hilb! This lecture will be dedicated to answering the following suprisingly subtle question:

Question: What is a *Unitary* TQFT?

Morally, we expect a unitarity TQFT to be an $\mathbb{E}_\infty$-functor from $(\mathrm{Bord}^X_{n,n-1}, \sqcup)$ into $(\mathrm{sHilb}, \otimes)$. sHilb should obviously have objects given by so-called **super Hilbert spaces**, i.e super vector spaces with a non-degenerate pairing that behaves well with the Koszul sign rule. But what should be the morphisms?

Example 2.1: Bad Definitions of Hilb

It seems that all of the natural choices lead to somehow ill-behaved TQFTs.

1. One natural choice would be to just take all linear maps as the morphisms in Hilb, but in this case the forgetful functor simply gives an equivalence $\mathrm{Hilb} \xrightarrow{\sim} \mathrm{Vect}$. This does not provide a resolution for unitarity in TQFTs, as such a structure really should provide us with more information than a regular TQFT.

[2] The intuition is that a principal G-bundle records how the fiber changes when transported around loops in the base. Since the fiber is a G-torsor, lifting a loop can return to the original fiber point multiplied by some group element $g \in G$; this is called monodromy. Choosing a different point in the fiber changes all monodromies by simultaneous conjugation. Thus the monodromy is a homomorphism $\rho : \pi_1(M) \to G$, well-defined up to conjugation.

2. Perhaps an even more natural choice would be to take isometries ($\subset$ unitaries) between Hilbert spaces as the set of morphisms. This in fact fails even worse than the previous example upon looking at the image of bordisms in this category of Hilbert spaces!

   The issue is that for every Y^{n-1}, we can take its cylinder bordism $Y^{n-1} \times [0, 1]$ and bend it to get a bordism from $Y \otimes \overline{Y} \to \emptyset$. The TQFT functor should then assign an **isometry** $Z(Y) \otimes Z(\overline{Y}) \to \mathbb{C}$.

   However, in order for this to be an isometry, no vectors can be mapped to $0 \in \mathbb{C}$, which is only possible if the domain is 1-dimensional. Thus, if we naively take our morphisms in Hilb to be isometries or unitarities, **every TQFT must assign a 1-dimensional state space to every single closed manifold,** making this definition completely nonsensical.

The primary reason these definitions fail is that **adjoints of morphisms should be an additional structure that is considered as part of the definition of our monoidal categories.** As of now, we just have looking at Hilb as a regular symmetric monoidal category and considering functors out of Bord which have no real knowledge of the relation between a linear map U and a linear map $U^\dagger$ between Hilbert spaces. To begin resolving this, we pose the following definition for a new type of category that encapsulates some of this structure.

Definition 2.2: †-Categories

A †-category[a] is a category $\mathcal{C}$ equipped with a functor $(\cdot)^\dagger : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ such that $\left((\cdot)^\dagger\right)^\dagger = id_{\mathcal{C}}$ and $x^\dagger = x$ for all objects $x \in \mathcal{C}$.[b]

Furthermore, a †-functor $F : (C, \dagger) \to (D, \dagger)$ between dagger categories is a functor of the underlying categories such that $F(c^\dagger) = F(c)^\dagger$. Thus, we can define the category $\mathrm{Cat}_1^\dagger$ to be the $(2, 1)$ category with objects †-categories, 1-morphisms †-functors, and invertible 2-morphisms given by unitary natural isomorphisms (in the sense that $\phi_x^\dagger = \phi_x^{-1}$ for all objects).

We then can define an $\mathbb{E}_\infty$-†-category to be an $\mathbb{E}_\infty$-monoid internal to $\mathrm{Cat}_1^\dagger$! This basically just means that all of the unitors, associators, and braiding natural isomorphisms are unitary.

[a] Pronounced as "dagger."

[b] If you are a category theory aficionado, perhaps demanding that this functor obeys these two identities on the nose disturbs you deeply. The key insight to get over this violation of the principle of equivalence is that **one should in fact not think of $(\cdot)^\dagger$ as a functor** in the same way that we do not think of the composition of morphisms as a functor with associativity enforced on the nose, but rather as an axiom defining what a category **is**. So too do we think of $(\cdot)^\dagger$ obeying the conditions above simply as the defining data of the natural notion of a †-category.

It is easy to see that Hilb with any kind of linear maps as morphisms is a †-category with $(\cdot)^\dagger$ just given by the adjoint of linear maps. With this structure now available to us, there is a clear path to potentially defining a unitary TQFT.

Idea: Define a †-structure on $\mathrm{Bord}_{n,n-1}^X$ and require that $Z : \mathrm{Bord}_{n,n-1}^X \to \mathrm{sHilb}$ is an $\mathbb{E}_\infty$-† functor!

Let's start by looking at some examples of tangential structures to see if we can gain some insight on how to define this dagger structure on an aribtrary X-structured bordism category.

The easiest example is obviously when $X = BO$. In this case, there is no tangential structure and bordisms are unoriented. Thus, by reversing the bordism in the sense of swapping the identification of ingoing and outgoing boundaries, we get a †-structure. This will clearly square to the identity on bordisms and leave the boundary manifolds unchanged, as they have no notion of orientation.

The next most natural tangential structure to consider is $X = BSO$, which yields the category of oriented bordisms. This case makes intuitive sense; consider the following two step process:

1. First, reverse the bordism as in the unoriented case shown in Fig. 2. However, in the oriented case, this alone does not act as the identity on objects. It instead maps a bordism $M : Y_1 \to Y_2$ into a new

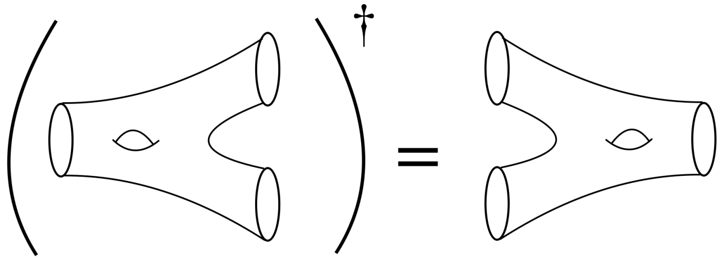


Figure 2: †-structure on Bord when $X = BO$.

bordism from $M^\vee : Y_2^\vee \to Y_1^\vee$, where $Y^\vee := Y$ but with the $X(n)$ structure on $TY \oplus \mathbb{R}$ composed with $id_{TY} \oplus -id_{\mathbb{R}}$.

2. We then orientation-reverse the entire bordism M by composing $M \to BSO$ with the $\mathbb{Z}_2$ action on BSO coming from the fact that BSO double covers BO.

This process ultimately fixes the boundary manifolds while reversing the orientation of the bordism itself, giving a well defined †-structure on $\mathrm{Bord}_{n,n-1}^{BSO}$. In fact, this is what Atiyah did in his original paper on TQFTs.

Remark 2.1: Unstably Framed Bordisms and †-structures

In general, this exact procedure will not work for general tangential structures beyond orientations. Perhaps surprisingly, even for the simplest *unstable* tangential structures, the structured bordism category can sometimes fail to admit any dagger structure at all.

For instance, let $X(n) = \mathrm{pt} \to BO(n)$ and consider the category $\mathrm{Bord}_{n,n-1}^{X(n)}$, where morphisms have an $X(n)$-structure, i.e a framing, and objects have a compatible $X(n-1)$ structure determined by the homotopy pullback $X(n-1) := * \times_{BO(n)} BO(n-1) = *$, which is just a framing of the boundary manifolds.

This category is just the category of framed bordisms. It is important to note that the tangential structures here are not necessarily coming from the pullback of a **stable** tangential structure. A stable framing would be a trivialization of $TM \oplus \underline{\mathbb{R}}^k$ for large enough k, which is a much easier condition to meet.

Now, consider the category $\mathrm{Bord}_{2,1}^{\mathrm{fr}} := \mathrm{Bord}_{2,1}^{X(2)}$. Suppose that this framed bordism category indeed had a †-structure. Then, for every framed bordism $M : Y_1 \to Y_2$, there is another framed bordism $M^\dagger : Y_2 \to Y_1$.

But consider the bordism given by a torus with a disk removed $M := T^2 \setminus D^2 : \emptyset \to S^1$. Since this space is an oriented 2d manifold with boundary, it can indeed be endowed with a framing. We can then take the dagger to get a bordism $M^\dagger : S^1 \to \emptyset$, where since the † acts as the identity on objects, we can glue M and $M^\dagger$ together along S^1. Then, since gluing preserves tangential structure, upon composing these bordisms $M^\dagger \circ M : \emptyset \to \emptyset$, we obtain a closed genus $g = 2$ surface with a framing.

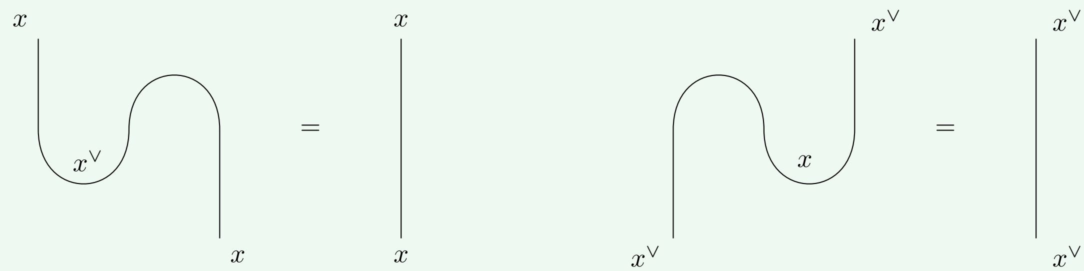
However, it is known that a non-vanishing Euler class is an obstruction to a framing, meaning that closed manifolds with genus $g \geq 2$ **cannot** admit a framing! Thus, we have reached a contradiction, meaning that $\mathrm{Bord}_{2,1}^{\mathrm{fr}}$ cannot admit a †-structure!

Even though our strategy for the case of oriented bordisms does not work in general, by examining the $X = BSO \to$ case, we did potentially learn something about a kind of process which may construct †-structures on stably X-structured bordism categories. Crucially, it is reasonable to expect that there should be some kind of two step process of separate $\mathbb{Z}_2$ actions which together give something which act trivially on objects to give a †-structure, even if each $\mathbb{Z}_2$ action individually may not define one. This can be thought of as some kind of CPT theorem for tangential structures.

Let us start by trying to axiomatize what we did in step 1 of constructing the †-structure for the oriented bordism category. We begin by defining the following structure on monoidal categories.

Definition 2.3: Duality

Let $\mathcal{C}$ be an $\mathbb{E}_\infty$ category[a] and $x \in \mathcal{C}$. A **dual** of x is another object $x^\vee$ together with morphisms $\mathrm{ev}_x : x^\vee \otimes x \to 1$ (represented pictorially as a "cap") and $\mathrm{coev}_x : 1 \to x \otimes x^\vee$ ("represented pictorially as a "cup") such that the so-called snake identities hold:



(18)

Note that a dual of an object is unique in the sense that if there is another dual $(x'^\vee, \mathrm{ev}', \mathrm{coev}'$, then there is a unique isomorphism $x^\vee \simeq x'^\vee$ which intertwines the evaluation and coevaluation maps.

We call a monoidal category **rigid** if every object admits a dual (i.e if it is **dualizable**).

[a] Really, $\mathcal{C}$ just needs to be monoidal, but we assumed it was $\mathbb{E}_\infty$ so we didn't need to worry about distinguishing between left and right duals.

There are some important properties of duals that will show up in our study of TQFTs. First, suppose we have an $\mathbb{E}_\infty$ functor $F : \mathcal{C} \to \mathcal{D}$ for $\mathcal{C}$ a rigid $\mathbb{E}_\infty$-category; then, for any object $x \in \mathcal{C}$, we have:

$$F(x) \otimes F(x^\vee) \simeq F(x \otimes x^\vee) \xrightarrow{F(\mathrm{ev})} F(1_\mathcal{C}) \simeq 1_\mathcal{D}. \tag{19}$$

Using similar reasoning for the coevaluation map and the snake identities, we can see that

$$F(x)^\vee \simeq F(x^\vee) \tag{20}$$

for all $x \in \mathcal{C}$. Thus, the image of a rigid category under a monoidal functor must be rigid.

Observe that $\mathrm{Bord}^X_{n,n-1}$ is a rigid $\mathbb{E}_\infty$-category, as for any object Y^{n-1}, we can take the cylinder bordism $Y \times [0,1]$ and bend it to get a bordism $\mathrm{ev} : Y \sqcup Y^\vee \to \emptyset$ and similarly for the coevaluation map, as seen below. Then, since the gluing of morphisms is defined only modulo diffemorphism, it is clear that if we glue

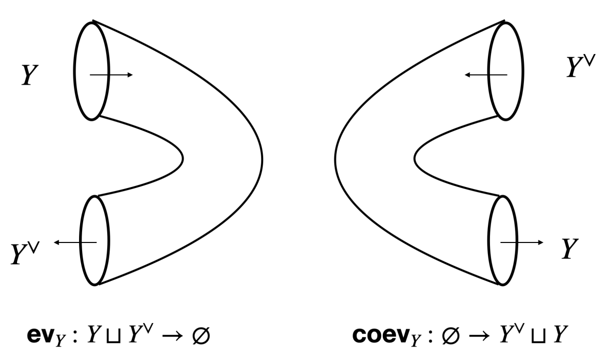


Figure 3: Rigidity of $\mathrm{Bord}^X_{n,n-1}$

together the evaluation and coevaluation maps, we can always straighten them to get a cylinder such that the snake identities hold.

With the above two facts, we can see that any TQFT $Z : \mathrm{Bord}^X_{n,n-1} \to \mathrm{Vect}$ must factor through the category of dualizable vector spaces. Now, we know that a vector space over $\mathbb{C}$ is dualizable if and only it is finite dimensional, meaning that any n-dimensional TQFT must assign only finite dimensional vector spaces to closed $(n-1)$-manifolds.

Definition 2.4: Dual Functor

Given a rigid monoidal category $\mathcal{C}$, we have the so-called **dual functor** $(\cdot)^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. This is a monoidal functor which maps any object to its dual object and any morphism $f : x \to y$ to another morphism $f : y^\vee \to x^\vee$ determined by the following universal property:

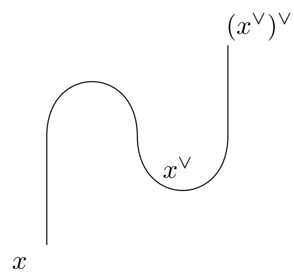
$$\begin{tikzcd} y^\vee \otimes x \arrow[r, "f^\vee \otimes id"'] \arrow[d, "id \otimes f"] & x^\vee \otimes x \arrow[d, "ev_x"'] \\ y^\vee \otimes y \arrow[r, "ev_y"] & 1 \end{tikzcd} . \tag{21}$$

Explicitly, this map is given by:

$$f^\vee : y^\vee \to y^\vee \otimes 1 \xrightarrow{\mathrm{coev}_x} y^\vee \otimes x \otimes x^\vee \xrightarrow{id \otimes f \otimes id} y^\vee \otimes y \otimes x^\vee \xrightarrow{\mathrm{ev}_y \otimes id} x^\vee . \tag{22}$$

If one draws the string diagram for this morphism, it follows easily from the snake identities that this choice of $f^\vee$ makes the above diagram commute.

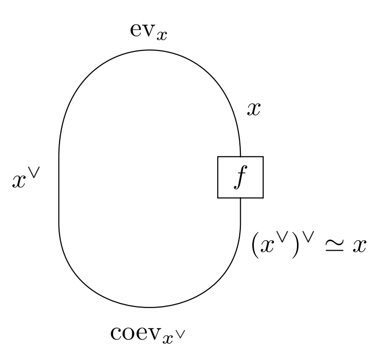
A rigid category $\mathcal{C}$ also admits a natural **pivotal structure,** i.e a natural isomorphism $(\cdot)^{\vee\vee} \simeq \mathrm{id}_{\mathcal{C}}$ given by:



. (23)

One can check that this is indeed natural with respect to maps $f : x \to y$ by playing with the graphical calculus of this map and the map $f^{\vee\vee} : (x^\vee)^\vee \to (y^\vee)^\vee$.

Using this pivotal structure, we can define a trace for morphisms $f : x \to x$ in $\mathcal{C}$ via:



. (24)

With this, we have developed the algebraic structure needed to formalize step 1 of constructing †-structures. The property of rigidity for a monoidal category should be viewed analogously to what we did for $\mathrm{Bord}^{\mathrm{BSO}}_{n,n-1}$ in the sense that taking the dual of a bordism simply amounts to reading it backwards.

It is important to note that just as was the case in our motivating example, $(\cdot)^\vee$ acts nontrivially on objects. Thus, we would like some other $\mathbb{Z}_2$ action $\overline{(\cdot)}$ — meant to be analogous to an orientation reversal of a bordism — such that the diagonal action $\overline{(\cdot)}^\vee$ acts nontrivially on objects and gives a †-structure.

Remark 2.2: $\mathbb{Z}_2$-actions on Categories

Suppose $\mathcal{C}$ is a rigid $\mathbb{E}_\infty$-category and consider a $\mathbb{Z}_2$ action $\overline{(\cdot)}$. Naively, we would like to describe this by a map $\mathbb{Z}_2 \to \mathrm{Aut}_\mathcal{C}$. But since $\mathrm{Aut}_\mathcal{C}$ sits inside the $(2,1)$ category $\mathrm{Cat}_1^{\mathbb{E}_\infty}$, it is not just a group, but in fact a **2-group!**[a]

Thus, the $\mathbb{Z}_2$ action on $\mathcal{C}$ is better described by a **weak 2-functor**[b], aka a **pseudofunctor**, $B\mathbb{Z}_2 \to Cat_1^{\mathbb{E}_\infty}$, which maps $* \to \mathcal{C}$.

Explicitly, this amounts to an $\mathbb{E}_\infty$-functor $\overline{(\cdot)} : \mathcal{C} \to \mathcal{C}$ associated to the nontrivial object $-1 \in \mathbb{Z}_2$ together with a natural $\mathbb{E}_\infty$-isomorphism $\overline{\overline{(\cdot)}} \overset{\eta}{\simeq} id_\mathcal{C}$ and the condition that $\overline{\eta_x} = \eta_{\overline{x}}$ for every object $x \in \mathcal{C}$ (coming from the associativity diagram for $g \cdot (g \cdot g) = (g \cdot g) \cdot g$ for the nontrivial object $g \in \mathbb{Z}_2$).

[a] A 2-group is a monoidal category where all objects have an inverse (up to isomorphism) with respect to $\otimes$ and all morphisms are invertible.

[b] A weak 2-functor $F : \mathcal{C} \to \mathcal{D}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a function $\mathrm{Ob}_\mathcal{C} \to \mathrm{Ob}_\mathcal{D}$ and a functor $F_{x,y} : \mathrm{Hom}(x,y) \to \mathrm{Hom}(F(x), F(y)$ for every hom-category, equipped with invertible 2-morphisms $\eta_x : id_{F(x)} \simeq F(id_x)$ and $\rho_{f,g} : F(g) \circ F(f) \simeq F(g \circ f)$ natural in $f : x \to y$ and $g : y \to z$ subject to coherence diagrams.

Let us briefly remark that even though the dual functor satisfies $(\cdot)^{\vee\vee} \simeq id_\mathcal{C}$, it does not give a $\mathbb{Z}_2$ action on $\mathcal{C}$, as it is a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ rather than a true endofunctor. However, it does give a $\mathbb{Z}_2$ action on the **core**[3] of the category, $\mathcal{C}^\simeq$, via:

$$\mathcal{C}^\simeq \xrightarrow{(\cdot)^\vee} (\mathcal{C}^{\mathrm{op}})^\simeq \xrightarrow{(\cdot)^{-1}} \mathcal{C}^\simeq, \tag{25}$$

where $(\cdot)^{-1}$ is the functor which inverts all morphisms (which is possible since we restricted to the maximal groupoid $\mathcal{C}^\simeq$).

Now, we'd like to know how a $\mathbb{Z}_2$-action $\overline{(\cdot)}$ interplays with the dual functor. Observe that $\overline{\mathrm{ev}_x} : \overline{x^\vee \otimes x} \simeq \overline{x^\vee} \otimes \overline{x} \to \overline{1} \simeq 1$ and $\overline{\mathrm{coev}_x} : \overline{1} \simeq 1 \to \overline{x \otimes x^\vee} \simeq \overline{x} \otimes \overline{x^\vee}$ since $\overline{(\cdot)}$ is an $\mathbb{E}_\infty$-functor. Thus, $\overline{\mathrm{ev}_x}$ and $\overline{\mathrm{coev}_x}$ give evaluation and coevaluation maps for $\overline{x}$ with dual $\overline{x^\vee}$, which respect the snake identities up to coherent isomorphism.

Then, since duals are unique up to isomorphism, we must have that $\overline{(\cdot)^\vee} \simeq (\overline{(\cdot)})^\vee$, meaning that $\overline{(\cdot)}^\vee$ is a well-defined $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on the core $\mathcal{C}^\simeq$.

Definition 2.5: Hermitian Construction

Let $\mathcal{C}$ be a rigid $\mathbb{E}_\infty$-category with a $\mathbb{Z}_2$ action $\overline{(\cdot)}$. **A Hermitian pairing on an object** $x \in \mathcal{C}$, is a $\mathbb{Z}_2$ fixed point under the diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, meaning that there is an isomorphism $x \overset{h_x}{\simeq} (\overline{x})^\vee$ satisfying $h_{\overline{x}^\vee} = \overline{h_x}^\vee$.

Equivalently, we could curry this to get a map $h : \overline{x} \otimes x \to 1$ such that the following diagram commutes.

$$\begin{tikzcd}
\overline{\overline{x} \otimes x} \arrow[rr, "\overline{h}"] \arrow[d, "\simeq"'] & & \overline{1} \arrow[r, "\simeq"] & 1 \\
\overline{\overline{x}} \otimes \overline{x} \arrow[r, "\simeq"'] & x \otimes \overline{x} \arrow[r, "\simeq"'] & \overline{x} \otimes x \arrow[ur, "h"'] &
\end{tikzcd} \tag{26}$$

[3] The core of $\mathcal{C}$, denoted $\mathcal{C}^\simeq$ is its maximal groupoid, obtained by throwing away non-invertible morphisms

With this data, observe that given a map $f : x \to y$ between two $\mathbb{Z}_2$ fixed points, we can define a map

$$f^\dagger : y \xrightarrow{h_y} \overline{y}^\vee \xrightarrow{\overline{f}^\vee} \overline{x}^\vee \xrightarrow{h_x^{-1}} x. \tag{27}$$

Thus, the category $\mathrm{Herm}(\mathcal{C}, \overline{(\cdot)})$ with objects $\mathbb{Z}_2$ fixed points $(x \in \mathcal{C}, h_x : x \simeq \overline{x}^\vee)$ and morphisms $(x_1, h_1) \to (x_2, h_2)$ given simply by morphisms $x_1 \to x_2$ in $\mathcal{C}$ is a †-category with the † defined above.

This category is called the **Hermitian completion** of $(\mathcal{C}, \overline{(\cdot)})$, and in fact most †-categories can be constructed this way. It's important to note that this category should not be thought of as some kind of "universal †-completion," but rather a restriction to objects which admit a Hermitian structure.

As an example, consider the category $\mathrm{Herm}_\mathbb{C}$ of Hermitian vector spaces, i.e vector spaces with a sesquilinear map $\langle \cdot, \cdot \rangle : \overline{V} \otimes V \to \mathbb{C}$ satisfying $\overline{\langle w, v \rangle} = \langle v, w \rangle$. Observe that we have

$$\mathrm{Herm}(\mathrm{Vect}_\mathbb{C}, \overline{(\cdot)}) = \mathrm{Herm}_\mathbb{C}, \tag{28}$$

where $\overline{(\cdot)}$ is the scalar complex conjugate, as one can check easily that the map $h : \overline{x} \otimes x$ together with its consistency diagram in Eq. 26 give exactly the sesquilinear map $\langle \cdot, \cdot \rangle$. Note however, that this $\mathrm{Herm}_\mathbb{C} \neq \mathrm{Hilb}_\mathbb{C}$, as there is no condition guaranteeing the positivity of this sesquilinear map.

With this, we have finally formalized the construction of a †-structure from two $\mathbb{Z}_2$-actions, which allows us to give the following definition.

Definition 2.6: Unitary TQFT

An X-structured **unitary** TQFT is an $\mathbb{E}_\infty$-†-functor

$$Z : \mathrm{Bord}^X_{n,n-1} \to \mathrm{sHilb}. \tag{29}$$

By the discussion above, since the bordism category is rigid, we know that giving a **reflection TQFT,** i.e a $\mathbb{Z}_2$ equivariant $\mathbb{E}_\infty$-functor:

$$\left(\mathrm{Bord}^X_{n,n-1}, \overline{(\cdot)}\right) \to \left(\mathrm{sVect}_\mathbb{C}, \overline{(\cdot)}\right), \tag{30}$$

in the sense that $Z(\overline{Y}) \simeq \overline{Z(Y)}$, is equivalent to giving a †-functor into $\mathrm{sHerm}_\mathbb{C}$.

Thus, if we are able to supply a positivity condition on the Hermitian pairings and find a $\mathbb{Z}_2$ action on $\mathrm{Bord}^X_{n,n-1}$, then we can construct a unitary TQFT.

To summarize, in this lecture we found that if one can construct a $\mathbb{Z}_2$ action $\overline{(\cdot)}$ on a rigid tensor category $\mathcal{C}$, then $\overline{(\cdot)}$ together with the dual $(\cdot)^\vee$ gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on the core $\mathcal{C}^\simeq$ which naturally provides a well-defined †-category upon passing to the Hermitian completion of $\mathcal{C}$!

3 Lecture 3: $\mathrm{Bord}^X_{n,n-1}$ is a †-Category

Equipped with the algebraic structures defined in the previous lecture, we now know how to define and construct a unitary TQFT from a few more tractable ingredients. From the Hermitian construction outlined in the previous lecture, we know that since bordism categories are rigid, we just need to find an $\mathbb{E}_\infty$ $\mathbb{Z}_2$ action $\overline{(\cdot)} \curvearrowright \mathrm{Bord}^X_{n,n-1}$ such that the induced diagonal $\mathbb{Z}_2$ action $\overline{(\cdot)}^\vee \curvearrowright \left(\mathrm{Bord}^X_{n,n-1}\right)^\simeq$ is trivial on objects. This construction will give us a dagger structure, and then to obtain a unitary TQFT, we will simply need to supply a positivity condition.

But what should the $\mathbb{Z}_2$ action $\overline{(\cdot)}$ be?

The initial idea: Motivated by the case of oriented bordisms, one may posit "$\overline{(\cdot)}$ is orientation reversal."

A more sophisticated idea: Observe that for an abelian group[4] A (like $\mathbb{Z}_2$), BA has a well-defined group operation, i.e a multiplication **functor** $m : BA \times BA \to BA$, an inverse functor $i : BA \to BA$, and a unit map which satisfy the usual properties we would expect for a group.

Thus, if $B\mathbb{Z}_2 \to \mathcal{C}$, $* \to x \in \mathcal{C}$ is a $\mathbb{Z}_2$-action on x, then from the group structure on $B\mathbb{Z}_2$, we get an induced $\mathbb{Z}_2 \times \mathbb{Z}_2$ action given by:

$$B\mathbb{Z}_2 \times B\mathbb{Z}_2 \xrightarrow{\circ} B\mathbb{Z}_2 \to \mathcal{C}. \tag{31}$$

Because of the nature of the multiplication map and the fact that $\mathbb{Z}_2$ is abelian, the generators $(g, 1)$ and $(1, g)$ will thus act the same on x and commute with each other. Since $\mathbb{Z}_2$ is abelian, we also have another $\mathbb{Z}_2$ action coming from the inverse: $g \cdot x := g^{-1}x$, but for $\mathbb{Z}_2$, we have $g^{-1} = g$.

Conclusion: For any A-action (with A an abelian group), there exists an induced $A \times A$ action and another *inverse* A-action coming from the group structure on BA. In particular, for $A = \mathbb{Z}_2$, the inverse action is the same as the original action, i.e it seems that $\overline{x}^\vee \simeq x$ should be the same as $\overline{x} \simeq x^\vee$.

With all of this in mind, we have the following list of things that we would like to do to construct a †-structure on a structured bordism category.

To do list:

1. Understand G-actions on $\mathrm{Bord}^X_{n,n-1}$
2. Understand the $\mathbb{Z}_2$ action of duals on $\left(\mathrm{Bord}^X_{n,n-1}\right)^{\simeq}$
3. Understand how these two actions commute
4. Find $\mathbb{Z}_2$ action $\overline{(\cdot)}$ such that $\overline{Y}^\vee \simeq Y$ for all $Y \in \mathrm{Bord}^X_{n,n-1}$, which we know from yesterday will give us a well-defined †-structure.

1. G-actions on $\mathrm{Bord}^X_{n,n-1}$

Let us start with item 1 on our list. First, observe that the assignment

$$(X \to BO) \mapsto \mathrm{Bord}^X_{n,n-1} \tag{32}$$

is functorial in the slice category $\mathcal{S}/BO$, where $\mathcal{S}$ is some category of spaces (e.g topological spaces, CW complexes, ∞-groupoids, anima, etc).

Recall that the slice category $\mathcal{S}/BO$ will have objects $X \to BO$ with morphisms given by maps $f : X \to Y$ which make the triangle diagram of maps into BO commute up to homotopy. Such a morphism f gives exactly the data needed to give a functor $\mathrm{Bord}^X_{n,n-1} \to \mathrm{Bord}^Y_{n,n-1}$ by transforming the X-structure on every object and morphism of the bordism category.

Thus, one way to describe a G-action on the X-structured bordism category is to answer the question of what is a G-action on $X \xrightarrow{\rho} BO$.

To answer this, we start by just looking at the action of G on a space using the unstraightening correspondence in topology. Recall that a group G may be regarded as a category with one object and morphisms given by elements of G, so that a functor

$$BG \xrightarrow{\alpha} \mathcal{S} \tag{33}$$

[4]Notice that this fails for a non-abelian group G as if we try to define the action of $m : BG \times BG \to BG$ by group composition then $m(f_1, g_1) \circ m(f_2, g_2) = f_1 \circ g_1 \circ f_2 \circ g_2$, while $m(f_1 \circ f_2, g_1 \circ g_2) = f_1 \circ f_2 \circ g_1 \circ g_2$, which means that in order for the multiplication m to be functorial, G must be abelian. Similarly, if we define the inverse $i : BG \to BG$ by just taking the inverse of morphisms, we have $i(f \circ g) = (f \circ g)^{-1} = g^{-1} \circ f^{-1}$ while $i(f) \circ i(g) = f^{-1} \circ g^{-1}$; thus, G must be abelian in order for i to be functorial as well.

is precisely the data of a space $\alpha(*) = F$ equipped with a coherent G-action. From such a functor α describing a G-action on F, we can form a fibration over BG with fiber F via:

$$\begin{array}{rc} \alpha(*) = & F \\ & \downarrow \\ \mathrm{colim}_{BG}(\alpha) = & EG \times_G F \\ & \downarrow \\ \mathrm{colim}_{BG}(\mathrm{const.}) = & BG \end{array} \quad . \tag{34}$$

[5]

Here colim is the so-called **homotopy colimit** of a functor. For the case of the functor α, its homotopy colimit is, roughly speaking, the unique space (up to homotopy) inside of which every n-simplex of n-composable group elements with vertices labeled by the space F can be filled by a coherent homotopy, i.e every diagram like this for all higher simplices should commute up to homotopy:

$$\begin{tikzcd} \mathrm{colim}_{BG}(\alpha) & F \arrow[l, dashed] \\ F \arrow[u, dashed] \arrow[ur] \arrow[r, "g_1"'] & F \arrow[u, "g_2"'] \arrow[ul, dotted] \end{tikzcd}$$

By the definition of the nerve construction of the universal bundle $EG \to BG$, we can see that the explicit realization of the colimit is

$$\mathrm{colim}_{BG}(\alpha) = EG \times_G F := (EG \times F)/\sim, \tag{35}$$

where $(x, f) \sim (gx, gf)$. This is a homotopy theoretic refinement of the notion of taking the quotient of a G-space which keeps track of all higher coherence information.

Conversely, given such a fibration over BG, loops in BG give rise to monodromy on the fiber F, which yields an action of $\Omega BG \simeq G$ on F. But this is exactly a functor $BG \to \mathcal{S}$ which sends the basepoint of BG to a space F.

We omit the exact details, but this correspondence between G-actions, or functors out BG, and fibrations over BG is the content of the so-called Grothendieck construction, which says that there is a categorical equivalence:

$$\boxed{\mathrm{Fun}(BG, \mathcal{S}) \simeq \mathcal{S}^{\mathrm{fib}}_{/BG}.} \tag{36}$$

Now that we have an understanding of G-actions on spaces in $\mathcal{S}$, we'd like to apply this to understand G-actions on tangential structures, i.e spaces over BO which are elements of the slice category $\mathcal{S}/BO$.

Given a tangential structure

$$X \xrightarrow{\rho} BO,$$

one can obtain a space with an O action via the **homotopy fiber** of ρ, which is defined by

$$\begin{tikzcd} \mathrm{fib}(\rho) \arrow[r, dashed] \arrow[d, dashed] & * \arrow[d] \\ X \arrow[ur, Rightarrow] \arrow[r, "\rho"'] & BO \end{tikzcd} \quad . \tag{37}$$

[5]Here, 'const.' is the constant functor taking the unique object of BG to the space consiting of a single point and all morphisms to the identity on that point.

Explicitly, denoting $(BO)^{[0,1]}$ as the space of maps from the interval into BO, this is given by the ordinary pullback of the diagram

$$\begin{tikzcd} & (BO)^{[0,1]} \arrow[d, "(\mathrm{src},\mathrm{targ})"] \\ X \times * \arrow[r, "(\rho,*)"'] & BO \times BO \end{tikzcd}, \tag{38}$$

which just means that the homotopy fiber consists of paths in BO between the image of the map ρ and the basepoint $*$:

$$\mathrm{fib}(\rho) = \{(x,\gamma) \in (X, (BO)^{[0,1]}) \,|\, \gamma(0) = \rho(x), \gamma(1) = *\}. \tag{39}$$

Then, a fibration over BO then gives an action of $\Omega BO \simeq O$ on the fiber $\mathrm{fib}(\rho)$ via monodromy.

Conversely, given a space F with an O-action, the Grothendieck construction gives us a fibration:

$$EO \times_O F \to BO. \tag{40}$$

It follows that a tangential structure $X \xrightarrow{\rho} BO$ is equivalent (in a homotopy theoretic sense) to a O-space via the correspondence:

$$\boxed{(X \xrightarrow{\rho} BO) \qquad \longleftrightarrow \qquad (O \curvearrowright \mathrm{fib}(\rho))} \tag{41}$$

Now, a G-action on a tangential structure is a functor

$$\alpha : BG \to \mathcal{S}_{/BO}$$

with $\alpha(*) = X \to BO$. This is exactly the data of a coherent G-action on the space X by automorphisms preserving the map to BO, i.e for every $g \in G$ there is a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "\rho"'] & X \arrow[d, "\rho"] \\ & BO. \end{tikzcd}$$

Unstraightening this using the Grothendieck construction defined above, we get a fibration over BG:

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d] & BO \\ \hat{X} := EG \times_G X \arrow[ur, dashed, "\widehat{\rho}"'] \arrow[d, "p"] & \\ BG & \end{tikzcd}, \tag{42}$$

where p is the projection onto $EG \to BG$ and the natural map $\widehat{\rho}$ is well-defined since ρ commutes with the G-action.

Thus, this data assembles into a space $\hat{X}$ over $BO \times BG \simeq B(O \times G)$ which pulls-back (restricts) the $O \times G$ action to the original fibration of $X \to BO$:

$$\begin{tikzcd} & \mathrm{fib}(\rho) \arrow[d] \\ X \arrow[r] \arrow[d, "\rho"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{X} \arrow[d, "(\hat{\rho},p)"] \\ BO \arrow[r, hook] & BO \times BG \end{tikzcd}. \tag{43}$$

Thus a G-action on the slice category $S_{/BO}$ is equivalent to an $O \times G$ action on the homotopy fiber of $\rho \in S_{/BO}$.

This construction is moreover functorial in the symmetry group G. Namely, suppose there is a homomorphism

$$\varphi : G_1 \to G_2, \tag{44}$$

such that the induced map on classifying spaces is compatible with the G_1 and G_2 actions:

$$\begin{tikzcd} BG_1 \arrow[dd, "B\varphi"'] \arrow[dr, "\alpha_1"] & \\ \arrow[r, phantom, "\Downarrow"] & \mathcal{S} \\ BG_2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}, \tag{45}$$

meaning that the G_1 action is simply the restriction of the G_2-action along the homomorphism φ. This is equivalent data to a pullback of fibrations:

$$\begin{tikzcd} \alpha_1(*) \arrow[d] \arrow[r, phantom, "\simeq"] & \alpha_2(*) \arrow[d] \\ EG_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG_2 \arrow[d] \\ BG_1 \arrow[r, "B\varphi"'] & BG_2 \end{tikzcd}. \tag{46}$$

Note that if we input $G_1 = O \hookrightarrow G_2 = O \times G$, then we exactly get the restriction of fibrations that we used in Eq. 43.

Thus, a map of symmetry groups corresponds geometrically to restriction, or pullback, of the associated family of tangential structures over the classifying space.

Example 3.1: Pullback of Stable Tangential Structure

We have seen this same concept before. Given a stable tangential structure $X \to BO$, we can pull it back to an unstable tangential structure $X(n)$ via the restriction to $BO(n)$:

$$\begin{tikzcd} X(n) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ BO(n) \arrow[r, hook] & BO \end{tikzcd}. \tag{47}$$

The moral of the story is that:

$$\boxed{G\text{-actions on tangential structures } X \xrightarrow{\rho} BO \quad \Longleftrightarrow \quad \text{Fibrations over } BO \times BG,}$$

where the fibrations over $BO \times BG$ restrict to the original tangential structure when pulled back along $BO \hookrightarrow BO \times BG$ via a general functoriality in G.

Then, since the assignment $X \xrightarrow{\rho} BO \to \mathrm{Bord}^X_{n,n-1}$ is functorial in the slice category $\mathcal{S}/BO$, these fibrations over $BO \times BG$ will induce a G-action on the X-structured bordism category.

We finish our discussion of general group actions on $\mathrm{Bord}^X_{n,n-1}$ by specializing to the case of an abelian group. Specifically, using the unstraightening discussion above, we see that an action of an abelian group A on a space $F \in \mathcal{S}$ can be expressed as a fibration over BA. Then, the group structure of BA gives us diagonal and inverse maps along which we can pullback to obtain new fibrations as follows:

$$\begin{tikzcd}
F \arrow[d] & & F \arrow[r, phantom, "\simeq"] \arrow[d] & F \arrow[d] & F \arrow[r, phantom, "\simeq"] \arrow[d] & F \arrow[d] \\
E \arrow[d, "p"] & \rightsquigarrow & E_\Delta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] & \tilde{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
BA & & BA \times BA \arrow[r, "+"] & BA & BA \arrow[r, "(\cdot)^{-1}"] & BA
\end{tikzcd} \tag{48}$$

where $E_\Delta := \{(e, b_1, b_2) \in E \times BA \times BA \,|\, p(e) = b_1 + b_2\}$ and $\tilde{E} := \{(e, b) \in E \times BA \,|\, p(e) = b^{-1}\}$.

2. $\mathbb{Z}_2$-Action of $\vee$ on $\left(\mathrm{Bord}^X_{n,n-1}\right)^\simeq$

Now that we understand group actions on $\mathrm{Bord}^X_{n,n-1}$ as fibrations over $BO \times BG$, we can start to specialize to understand the two desired $\mathbb{Z}_2$ actions which will give us a $\dagger$-structure. Recall that $O(1) = \mathbb{Z}_2$ and consider the $O(1)$ action on $X \xrightarrow{\rho} BO$ given by restricting the corresponding O-action along

$$BO \times BO(1) \hookrightarrow BO \times BO \xrightarrow{\oplus} BO. \tag{49}$$

This determines a pullback of fibrations, which upon pulling back one more time along $BO \hookrightarrow BO \times BO(1)$ encodes an $O(1)$ action on $X \xrightarrow{\rho} BO$:

$$\begin{tikzcd}
& X \times BO(1) \arrow[d, phantom, "\wr"] \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, bend right, "{\left[\begin{smallmatrix} \rho & \ominus\mathrm{incl} \\ 0 & \mathrm{id} \end{smallmatrix}\right]}"'] & \\
X \arrow[r] \arrow[d, "\rho"] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{X}_{tan} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\rho"] \arrow[ul, dashed, bend right] \\
BO \arrow[r, hook, "i_1"] & BO \times BO(1) \arrow[r, "\oplus"] & BO \arrow[l, dashed, bend left]
\end{tikzcd} . \tag{50}$$

Here, the matrix $\widehat{\rho}$ on the vertical red arrow denotes the map $(x, L) \in X \times BO(1) \mapsto (\rho(x) \ominus L, L)$[6]. From reading this diagram carefully, we can see a few important details.

1. First, given a stable bundle $V \to Y$, an $\widehat{X}_{tan}$ structure is described by a map from Y to $\widehat{X}_{tan}$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd}
& \hat{X}_{tan} \arrow[dr, "\hat{\rho}"] & \\
Y \arrow[ur] \arrow[rr, "V"'] & & BO
\end{tikzcd} . \tag{51}$$

   By the universal property of pullbacks, a map $Y \to \hat{X}_{tan}$ is determined by a map $Y \to BO \times BO(1)$, which assigns a stable vector bundle $V' \to Y$ and a line bundle $L \to Y$ together with a map $x : Y \to X$

[6] Here, $x \ominus L := x \oplus L^{-1}$, where L^{-1} really means the image of $L^{-1} \in BO(1)$ inside of BO

such that $\rho \circ x \simeq V' \oplus L$. The condition that the diagram in Eq. 51 commutes simply means that $V' \simeq V$.

Thus, an $\widehat{X}_{\mathrm{tan}}$-structure on a stable bundle $V \to Y$ consists of the data of a line bundle $L \to Y$ together with an X-structure on $V \oplus L$.

2. Next, from the splitting denoted by the blue dotted arrows, we can see that if we are given an X-structure on a stable bundle $V \to Y$, there is an induced $\hat{X}_{tan}$-structure on V given by the corresponding X-structure on $V \oplus \underline{\mathbb{R}}$.

3. Thus, $O(1) = \mathbb{Z}_2$ acts on an $X \xrightarrow{\rho} BO$ structure by first stabilizing with a single copy of $\underline{\mathbb{R}}$ and then reflecting it.

Looking at the induced $\mathbb{Z}_2$ action of bullet point 3 on the X-structured bordism category $\mathrm{Bord}^X_{n,n-1}$, we see that this action of stabilizing with a trivial bundle and then reflecting it exactly agrees with taking the dual $Y \to Y^\vee$ on objects. In conclusion, we have found that $\hat{X}_{tan}$ which we defined through pullbacks of fibrations exactly describes the $\mathbb{Z}_2$ action of $(\cdot)^\vee$ on $\left(\mathrm{Bord}^X_{n,n-1}\right)^{\simeq}$ defined by bending cylinders!

Now let us look at some explicit examples of $\hat{X}_{tan}$ to better understand the topological structure of $(\cdot)^\vee$.

Example 3.2: $\mathbb{Z}_2$-Action for $X = BSO$

Consider the case of an orientation structure, $X = BSO$. In this case, we obtain an $O(1)$ action via:

$$\begin{tikzcd} \hat{X}_{tan} \simeq BO \arrow[r, "(\cdot)\oplus\det(\cdot)"] \arrow[d, "(\mathrm{id}, \det)"'] \arrow[dr, phantom, "\lrcorner", very near start] & BSO \arrow[d] \\ BO \times BO(1) \arrow[r, "\oplus"] & BO \end{tikzcd}, \tag{52}$$

where $\det : BO \to BO(1)$ is the induced map on classifying spaces corresponding to the determinant homomorphism and the vertical map from BSO to BO simply forgets the orientation.

Observe that $V \oplus \det(V)$ indeed lands in BSO as an oriented vector bundle is just a trivialization of the determinant line bundle and $\det(V \oplus \det(V)) = \det(V) \otimes \det(\det(V)) = \det(V) \otimes \det(V)$, and the tensor product of a line bundle with itself is always trivializable.

Hence, we can indeed see that a $\hat{X}_{tan} \simeq BO$-structure on a bundle $V \to Y$ is a line bundle $L \simeq \det(V) \to Y$ and an orientation on $V \oplus \det(V)$, which matches the general discussion above.

Example 3.3: $\mathbb{Z}_2$-Action for $X = BSpin$

Now consider the more complicated example of $X = BSpin$. In this case, $\hat{X}_{tan}$ is defined by the following pullback

$$\begin{tikzcd} \hat{X}_{tan} \simeq BPin^- \arrow[r, dashed, "(\cdot)\oplus\det(\cdot)"] \arrow[d, dashed, "(\mathrm{id},\det)\circ\mathrm{hofib}(w_1^2+w_2)"'] \arrow[dr, phantom, "\lrcorner", very near start] & BSpin \arrow[d, "\mathrm{hofib}(w_1,w_2)"] \\ BO \times BO(1) \arrow[r, "\oplus"] & BO \end{tikzcd}. \tag{53}$$

We computed this pullback using the following reasoning. Recall that a bundle $V \to Y$ admits a spin structure if and only if $w_1(V) = w_2(V) = 0$, i.e $BSpin$ is the homotopy fiber of the map $(w_1, w_2) : BO \to K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2)$. Then, bundles V with $\hat{X}_{tan}$-structure should be such that $V \oplus \det(V)$ admit a spin structure, i.e its first and second Stiefel-Whitney classes vanish. Observe that the first Stiefel-Whitney class of this sum vanishes, as

$$w_1(V \oplus \det(V)) = w_1(V) + w_1(\det(V)) = w_1(V) + w_1(V) = 0, \tag{54}$$

where $w_1(V) = w_1(\det(V))$ as $w_1(V)$ is defined exactly by pulling back along det. Then, we get the following constraint coming from the second Stiefel-Whitney:

$$w_2(V \oplus \det(V)) = w_2(V) + w_1(V) \cup w_1(\det(V)) + w_2(\det(V)) = w_2(V) + w_1(V) \cup w_1(V), \tag{55}$$

where we used the fact that $w_i(\det(V)) = 0$ for $i > 1$ since it is just a line bundle. Thus, $\hat{X}_{tan}$ classifies exactly those vector bundles with $w_1^2 + w_2 = 0$, which is exactly the obstruction that needs to vanish for a bundle to admit a Pin^- structure!

3: Commuting $\mathbb{Z}_2$-Actions

Now, we have all of the structure we need to describe our desired $\overline{(\cdot)}^\vee$ action on $\mathrm{Bord}^X_{n,n-1}$.

Let $\widehat{X}_G \to BO \times BG$ be a G-action on $X \to BO$. Then, a G-action that commutes with duals is encoded by the data of the $G \times O(1)$-action given by restricting the $G \times O$-action along the direct sum map, just as we did in the previous section. In particular, pulling back the G-action along $O \times O(1) \xrightarrow{\oplus} O$ via

$$\begin{tikzcd} \hat{X}_{(tan,G)} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{X}_G \arrow[d] \\ BO \times BO(1) \times BG \arrow[r, "{(\oplus, \mathrm{id}_{BG})}"'] & BO \times BG \end{tikzcd} \tag{56}$$

gives a $G \times O(1)$ action on $X \to BO$, meaning that the G-action commutes with taking duals.

Remark 3.1: The Case of $\overline{(\cdot)} := (\cdot)^\vee$

To this end, the fact that the $\mathbb{Z}_2$-action $\widehat{X}_{tan}$ of taking duals commutes with itself is governed by the following pullback:

$$\begin{tikzcd} \hat{X}_{tan^2} \arrow[r, dashed] \arrow[d, dashed] & \hat{X}_{tan} \arrow[r] \arrow[d] & X \arrow[d] \\ BO \times BO(1) \times BO(1) \arrow[r, "{(\oplus, \mathrm{id})}"] & BO \times BO(1) \arrow[r, "\oplus"] & BO \end{tikzcd} . \tag{57}$$

Thus, the diagonal action $\overline{(\cdot)}^\vee$ in this case where $\overline{(\cdot)} = (\cdot)^\vee$ is governed by the image of the sum of two line bundles in BO; this is encoded by the map:

$$BO(1) \xrightarrow{\Delta} BO(1) \times BO(1) \hookrightarrow BO \times BO \xrightarrow{\oplus} BO. \tag{58}$$

Observe that, after passing to the relevant 2-type, an O-action on a 1-category $\mathcal{C}$ is encoded by a functor

$$\pi_{\leq 2} BO \simeq B\mathbb{Z}_2 \times B^2\mathbb{Z}_2 \longrightarrow \mathrm{Cat}.$$

The $B\mathbb{Z}_2$ factor records an involutive autoequivalence of $\mathcal{C}$, while the $B^2\mathbb{Z}_2$ factor records an involutive natural automorphism

$$(-1)^F : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}.$$

On the other hand, the group operation on BO induces a product $(w_1, w_2) \cdot (w_1', w_2') = (w_1 + w_1', w_2 + w_1 w_1' + w_2')$ on the 2-type $\pi_{\leq 2} BO$, where w_1 and w_2 are the first and second Stiefel-Whitney classes. Thus, the diagonal action $BO(1) \xrightarrow{\Delta} BO(1) \times BO(1) \to BO \times BO \xrightarrow{\oplus} BO$ gives

$$(w_1, 0)(w_1, 0) = (w_1 + w_1, w_1^2) = (0, w_1^2). \tag{59}$$

Since pulling back along the diagonal map produces the nontrivial class w_1^2, the diagonal $BO(1) \xrightarrow{\Delta} BO(1) \times BO(1) \xrightarrow{\oplus} BO$ cannot be null-homotopic. **Equivalently, the operation of taking duals twice is not canonically trivial at the level of tangential structures.**

To interpret this categorically, notice that the class w_1^2 lies in the $B^2\mathbb{Z}_2$-factor of the 2-type $\pi_{\leq 2} BO \simeq B\mathbb{Z}_2 \times B^2\mathbb{Z}_2$.

Under the correspondence between O-actions and functors out of this 2-type, the factor $B^2\mathbb{Z}_2$ records natural automorphisms of the identity functor. Thus, while the double-dual action is equivalent to the identity, there is no preferred choice of such an equivalence. Rather, the possible trivializations form a $\mathbb{Z}_2$-torsor whose difference is measured by the nontrivial natural automorphism

$$(-1)^F : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}.$$

In this sense, the class w_1^2 is the obstruction to canonically identifying the diagonal action with the identity. The diagonal is not trivial, but only trivial up to a choice, and changing this choice amounts precisely to composing with $(-1)^F$.[a].

The moral of the story here, is that even though the double dual acts as the identity on objects as a functor, there is additional coherence data encoded by w_1^2 which says that this double dual does not define a dagger structure as it is not involutive on the nose.

[a] The notation $(-1)^F$ is intentionally suggestive of familiar phenomena in physics. For example, a time-reversal symmetry T need not square canonically to the identity; instead, the possible identifications of T^2 with the identity may differ by fermion parity. The discussion above should be viewed as a homotopy-theoretic shadow of this phenomenon: the nontrivial class w_1^2 records a residual $\mathbb{Z}_2$ ambiguity in trivializing the square of the reflection action.

4. $\overline{(\cdot)}^{\vee}$ and a †-Structure on $\mathbf{Bord}^X_{n,n-1}$

Recall that we want to find a $\mathbb{Z}_2$-action $\overline{(\cdot)}$ on tangential structure, and hence on structured bordism categories, such that $\overline{(\cdot)}^{\vee}$ acts trivially on objects and squares to the identity on the nose to give a dagger structure.

We have argued that given a $\mathbb{Z}_2$-action $\overline{(\cdot)}$, the induced diagonal $\overline{(\cdot)}^{\vee}$ action is encoded by pulling back the corresponding fibration $X \to BO \times BO(1)$ along $BO \times BO(1) \times BO(1) \xrightarrow{(\oplus, \mathrm{id})} BO \times BO(1)$. Furthermore, that this diagonal action is canonically trivializable, and hence gives a †-structure, is the statement that the composition of horizontal maps in the pullback diagram is null-homotopic (which was not the case for the example of $\overline{(\cdot)} = (\cdot)^{\vee}$ in the previous section).

All of this motivates defining $\mathbb{Z}_2$-action $\overline{(\cdot)}$ via the following pull-back:

$$\begin{tikzcd} \widehat{X}_{nor} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\rho"] \\ BO \times BO(1) \arrow[r, "\ominus"] & BO \end{tikzcd} . \tag{60}$$

The diagonal action is then given by restricting once more to obtain:

$$\begin{tikzcd} \widehat{X}_{diag} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] && \widehat{X}_{nor} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\rho"] \\ BO \times BO(1) \times BO(1) \arrow[rr, "(\oplus, \mathrm{id})"] && BO \times BO(1) \arrow[r, "\ominus"] & BO \end{tikzcd} . \tag{61}$$

Notice that the composition

$$BO(1) \xrightarrow{\Delta} BO(1) \times BO(1) \hookrightarrow BO \times BO \xrightarrow{\ominus} BO, \tag{62}$$

which encodes the diagonal $\overline{(\cdot)}^{\vee}$ action, is by construction null-homotopic. Hence, $\overline{(\cdot)}^{\vee} \simeq \mathrm{id}_{\mathcal{C}}$ acts trivially on objects so that upon passing from an action on $X \xrightarrow{\rho} BO$ to an action on $\mathrm{Bord}^X_{n,n-1}$, we get a well-defined †-structure!

4 Lecture 4: Invertible Field Theories and Stable Homotopy

It is time for us to take a break from the subtle and technical formalism we had to introduce in the previous lecture and restrict to a very specific subset of TQFTs, known as invertible TQFTS (iTQFTs), which are particularly nice because of their amiability to the computational techniques of stable homotopy theory.

Let us start with some definitions.

Definition 4.1: Picard Groupoid

A **Picard groupoid** is a symmetric monoidal ($\mathbb{E}_\infty$) category $\mathcal{C}$ such that all morphisms are invertible with respect to composition and all objects are invertible with respect to $\otimes$, i.e for every $x \in \mathcal{C}$ there exists an object $x^{-1} \in \mathcal{C}$ such that $x \otimes x^{-1} \simeq 1$.

Now, given an arbitrary $\mathbb{E}_\infty$-category $\mathcal{C}$, we define $\mathrm{Pic}(\mathcal{C})$ to be its **maximal Picard groupoid.** This just means that we restrict to the collection of invertible objects and then take the maximal subgroupoid on these objects.

Equipped with this notion, we can give a precise definition of invertible TQFTs.

Definition 4.2: Invertible TQFT

An **invertible TQFT (iTQFT)** is a TQFT $Z : \mathrm{Bord}^X_{n,n-1} \to \mathrm{sVect}$ which factors through Pic(sVect), i.e there exists a functor $Z_i : \mathrm{Bord}^X_{n,n-1} \to \mathrm{Pic(sVect)}$ such that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Pic(sVect)} \arrow[dr, dashed] & \\ \mathrm{Bord}^X_{n,n-1} \arrow[ur, dashed, "Z_i"] \arrow[rr, "Z"'] && \mathrm{sVect} \end{tikzcd} \tag{63}$$

.

Since the unit object of sVect is $\mathbb{C}^{1|0}$, Pic(sVect) will consist of the collection of even and odd 1-dimensional super vector spaces (super lines) with automorphisms given by $\mathbb{C}^*$. Thus, an invertible theory will just assign 1d supervector spaces to Y^{n-1} and a non-zero complex number to bordisms.

Remark 4.1: Why iTQFTs?

Invertible QFTs seem like boring and trivial theories, but they are incredibly rich mathematical objects which are fundamental to twenty-first-century physics. While there is not currently a universally accepted theorem making the correspondence precise[a], it is widely believed that sufficiently local gapped quantum systems admit a topological field theory description in the infrared. The idea is that if one starts with a theory with some finite energy gap, as you run the renormalization group and look at longer and longer distances, i.e lower and lower energy scales, the gap will effectively become infinite. At energies much lower than the gap, local excitations become inaccessible, leaving only global topological information visible in the infared.

Furthermore, there are experimentally observed gapped systems which are completely trivial in the bulk (in the sense that there is a non-degenerate ground state and a lack of quasi-particles) but can respond nontrivially when their underlying symmetry or topology is probed by background gauge fields or boundary conditions. These phases are called symmetry-protected topological phases, as they can be deformed to a trivial theory if and only if the symmetry is neglected, and since these phases possess a unique ground state on every closed spatial manifold and no nontrivial quasiparticle excitations, the associated low-energy TQFT must be invertible.

Lastly, ‘t Hooft anomalies of d-dimensional quantum systems, which measure the failure of a symmetry to be realized in a completely local and gauge-invariant way, are naturally described by a (d+1)-dimensional invertible field theory. As we will see, iTQFTs have a rigid classification which implies that ‘t Hooft anomalies provide robust, discrete data about a theory that must remain the same as we look at the physics at different energy scales; in this sense, ‘t Hooft anomalies and iTQFTs are very powerful tools which can tell physicists a lot about the possible phases of a given system.

[a] The primary difficulty is that a finite energy gap is an analytic notion defined in terms of the spectrum of a Hamiltonian, while TQFTs are algebraic objects which make no reference to a Hamiltonian description. One in principle needs to find some suitable algebraic or topological definition of what it means for a theory to be gapped hich somehow abstracts the most fundamental properties of gapped systemswithout reference to a fixed Hamiltonian.

It is very important to note that we have the following pairs of adjoint functors between the category of Picard groupoids and the category of $\mathbb{E}_\infty$-categories:

$$\begin{tikzcd} \mathrm{PicGrpd} \arrow[r, hook] & \mathrm{Cat}_1^{\mathbb{E}_\infty} \arrow[l, bend right=50, "\mathrm{Pic}(\cdot)"', "\perp"] \arrow[l, bend left=50, "L", "\perp"'] \end{tikzcd} \quad . \tag{64}$$

Here, the functor L, stands for the *Picard localization* functor, which freely *adjoins* tensor inverses to every object and composition inverses to every morphism. Note that this is different from $\mathrm{Pic}(\cdot)$ which *restricts* to the maximal Picard groupoid of a symmetric monoidal category. Crucially, these adjoints give us a more concrete way of studying iTQFTs, i.e functors from bordism categories into Picard groupoids; specifically, for an $\mathbb{E}_\infty$-category $\mathcal{C}$, these adjunctions tells us that:

$$\mathrm{Fun}_{\mathrm{Cat}_1^{\mathbb{E}_\infty}}(\mathrm{Bord}_{n,n-1}^X, \mathrm{Pic}(\mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{PicGrpd}}(\mathrm{LBord}_{n,n-1}^X, \mathrm{Pic}(\mathcal{C})) \tag{65}$$

This tells us that instead of trying to construct full symmetric monoidal functors out of the X-structured bordism category, it is enough to just look at maps of Picard groupoids out of an enlarged localized bordism category. While this seems like abstract nonsense, this is a very powerful statement as algebraic topologists have spent the last seventy-odd years understanding how to compute spaces of maps of (higher) Picard groupoids: this is simply stable homotopy theory. Thus, by using this adjunction, we open ourselves up to lots of powerful tools making it possible to explicitly classify.

Remark 4.2: The Homotopy Hypothesis

Perhaps the most fundamental perspective in modern homotopy theory is the so-called **homotopy hypothesis**; roughly, this is the assertion of the following correspondence:

$$\infty-\text{groupoids} \quad \leftrightarrow \quad \text{spaces}. \tag{66}$$

An ∞-groupoid is a higher category with morphisms in every positive integer degree (i.e morphisms between objects, 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, and so on) such that every morphism in each of the levels is invertible.

An ∞-groupioid can be presented in terms of a simplicial decomposition with n-simplices given by compositions of $(n-1)$-morphisms as faces with the interior filled by n-morphisms encoding coherences; this is the so-called Kan complex presentation. From such combinatorial data, we can geometrically realize it as a space built out of simplices glued together according to the coherence data of the ∞-groupoid.

Conversely, given a space X, we can form an ∞-groupoid via the fundamental ∞-groupoid $\Pi_\infty(X)$ construction, which has objects points in X, morphisms given by paths between points, homotopies as 2-morphisms, and homotopies between homotopies encoding the higher morphisms.

If we truncate this correspondence and just look at regular 1-categories, we find a correspondence:

$$\text{1-groupoids} \quad \leftrightarrow \quad \text{1-types}, \tag{67}$$

where an n-type is a space X with $\pi_k(X) = 0$ for $k > n$. A 1-type X is classified by a set $\pi_0(X)$ and a group $\pi_1(X, x_0)$ for $x_0 \in \pi_0(X)$; under the equivalence of 1-types and groupoids, $\pi_0(X)$ corresponds to the set of isomorphism classes of objects in the groupoid and $\pi_1(X, x_0)$ corresponds to the group $\mathrm{Aut}(x_0)$.

Picard-groupoids, which we care about for the sake of classifying iTQFTs, are a special type of groupoid with a symmetric monoidal structure which is invertible up to isomorphism. This additional symmetric braiding structure on the groupoid leads to an even richer classification; in particular, given a Picard-groupoid $\mathcal{C}$, the symmetric monoidal structure allows us to infinitely deloop to obtain a connective spectrum[7] $\mathcal{P}$, $B\mathcal{P}$, $B^2\mathcal{P}$, $\cdots$ with two non-zero homotopy groups. This gives the space $|\mathcal{P}|$ the structure of an infinite loop space, making it a **stable 1-type.**

While it is useful to know that $\mathcal{P}$ fits into an infinite loop space for discussing deeper relations of iTQFT with stable homotopy, for the moment, it is just important to know what additional algebraic data goes into classifying Picard groupoids as opposed to regular groupoids. The invertible symmetric monoidal structure means that the set of isomorphism classes of objects $\pi_0(\mathcal{P})$ now has the structure of an **abelian group.** Additionally, the ability to deloop $\mathcal{P}$ infinitely tells us that $\pi_1(\mathcal{P}) = \mathrm{Aut}(1)$ must also be an **abelian group.** If we restrict to data that is visible at the skeletal level of isomorphism classes of objects, the only remaining information is encoded in how the braiding β of objects induces automorphisms.

Namely, observe that for any object $x \in \mathcal{P}$, the invertible symmetric monoidal structure gives an isomorphism $\mathrm{Aut}(x) \simeq \mathrm{Aut}(1)$ via:

$$f \in \mathrm{Aut}(x) \mapsto \mathrm{tr}(f) := \left(1 \simeq x \otimes x^{-1} \xrightarrow{f \otimes \mathrm{id}} x \otimes x^{-1} \simeq 1 \in \mathrm{Aut}(1)\right) \tag{68}$$

and

$$u \in \mathrm{Aut}(1) \mapsto \left(x \simeq 1 \otimes x \xrightarrow{u \otimes \mathrm{id}} 1 \otimes x \simeq x \otimes x \in \mathrm{Aut}(x).\right) \tag{69}$$

With this in mind, consider the map of a representative object to its self braiding $x \mapsto \beta_{x,x}$ for any $x \in \pi_0(\mathcal{P})$. The hexagon identity for braided monoidal categories tells us exactly that for $A, B, C \in \pi_0(\mathcal{P})$, we have

[7] A spectrum is a collection of spaces $\{X_n\}$ such that $X_n \simeq \Omega X_{n+1}$, where Ω is the based loop space functor. The homotopy groups of a spectrum are defined as $\pi_n(X) = \lim_{k\to\infty} \pi_{n+k} X_k$.

$\beta_{A,B\otimes C} = (\mathrm{id}_B \otimes \beta_{A,C}) \circ (\beta_{A,B} \otimes \mathrm{id}_C)$ and $\beta_{A\otimes B,C} = (\beta_{A,C} \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \beta_{B,C})$. This means that $x \otimes y \mapsto \beta_{x\otimes y, x\otimes y} = (\mathrm{id}_x \otimes \beta_{x\otimes y,y}) \circ (\beta_{x\otimes y,x} \otimes \mathrm{id}_y) = (\mathrm{id}_x \otimes \beta_{x,y} \otimes \mathrm{id}_y) \circ (\beta_{x,x} \otimes \beta_{y,y}) \circ (\mathrm{id}_x \otimes \beta_{y,x} \otimes \mathrm{id}_y)$.

But then since $\pi_1(\mathcal{P}) = \mathrm{Aut}(1)$ is abelian, when we look at the image of $\beta_{x\otimes y}$ in $\pi_1(\mathcal{P})$, the first and last factors combine to give $\mathrm{id}_x \beta_{x,y} \circ \beta_{y,x} \otimes \mathrm{id}_y$ which equals the identity since the braiding is symmetric. This means that $x \otimes y \mapsto \beta_{x,y} = \beta_{x,x} \otimes \beta_{y,y} \in \mathrm{Aut}(1)$, implying that this mapping is a homomorphism. That this is a homomorphism further implies that $x \otimes x \mapsto \beta_{x,x} \otimes \beta_{x,x} = \mathrm{id} \in \mathrm{Aut}(1)$ since the braiding is symmetric, meaning that this homomorphism factors through $\pi_0(\mathcal{P})/(2\pi_0(\mathcal{P}))$.

This can be summarized by the following classification theorem:

Theorem 4.3: Classification of Picard Groupoids (Sính)

A Picard groupoid $\mathcal{P}$ is classified uniquely by the following data:

1. An abelian group $A := \pi_0(\mathcal{P})$
2. Another abelian group $B := \pi_1(\mathcal{P}, 1)$
3. A group homomorphism $K : A/2A \to B$ defined by $x \mapsto \beta_{x,x} \in \mathrm{Aut}(x \otimes x) \simeq \mathrm{Aut}(1) = B$, which is referred to as the K-invariant.

The only difficulty beyond the argument we have provided above lies in showing that every Picard groupoid is equivalent to its skeletization; the proof of which can be found in Sính's thesis[8]

Example 4.4: Invariants of Simple Picard Groupoids

1. As an example, consider the case of $\mathcal{P} = \mathrm{Pic}(\mathrm{Vect})$. In this case, we have $\pi_0(\mathcal{P}) = 1$, $\pi_1(\mathcal{P}) = \mathbb{C}^*$, and a trivial K-invariant (as this category has trivial braiding).
2. As another example, consider $\mathcal{P} = \mathrm{Pic}(\mathrm{sVect})$. In this case, we have $\pi_0(\mathcal{P}) = \mathbb{Z}_2$ (odd/even superline), $\pi_1(\mathcal{P}) = \mathbb{C}^*$, and a homomorphism $K : \mathbb{Z}_2 \to \mathbb{C}^*$ defined by braiding the odd superline with itself $v \otimes v \xmapsto{\beta_{v,v}} -v \otimes v$.

So now we know how to compute iTQFTs, so long as we can perform the *simple* task of computing the localization of an X-structured bordism category! Luckily, there are some general techniques we can introduce to help compute and understand the 0'th and 1'st homotopy groups of a localized bordism category.

Let $\mathcal{C}$ be an $\mathbb{E}_\infty$-category. It is easy to see that the localization $L\mathcal{C}$ will have isomorphism classes of objects given by:

$$\pi_0(L\mathcal{C}) = \mathrm{ob}(\mathcal{C}) / \left(x \sim y \leftrightarrow \exists \text{ a morphism } f : x \to y\right). \tag{70}$$

This relation is not in general an equivalence relation, however, as it is not necessarily symmetric. But everything works out so long as we demand that $\mathcal{C}$ obeys the following stupid condition:

Definition 4.5: Reversible

A category $\mathcal{C}$ is **reversible** if $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq 0$ implies that $\mathrm{Hom}_{\mathcal{C}}(y, x) \neq 0$ for all $x, y \in \mathcal{C}$.

[8]Hoàng Xuân Sính was a Vietnamese mathematics teacher who recorded notes for Alexander Grothendieck's famous lectures in Hanoi during the American war. Having made an impression on Grothendieck, he agreed to advise her PhD thesis via correspondence throughout the course of the war. Famously, Sính and Grothendieck only sent a total of five messages to each other throughout the course of her dissertation, which developed the foundational theory of 2-groups: she is quoted as saying "If I remember correctly, he wrote to me twice and I wrote to him three times. The first time he wrote to me was to give me the subject of the thesis and the work plan; the second is to tell me to drop the problem of inverting objects if I can't do it. As for me, I think I wrote to him three times: the first was to tell Grothendieck that I couldn't invert objects because of the non-strict commutativity; the second is to tell him that I succeeded in inverting objects; and the third is to tell him that I have finished the job. The letters had to be very short because we were in time of war, eight months for a letter to arrive at its destination between France and Vietnam. When I finished my work in writing, I sent it to my brother, who lives in France, and he brought it to Grothendieck."

Next, observe that since localization simply formally adjoins all inverse arrows to $\mathcal{C}$ paths in $L\mathcal{C}$ will be given by zig-zags of morphisms in $\mathcal{C}$; the idea being that inside of $L\mathcal{C}$, we can traverse either forwards and backwards across any morphism.

For example, given objects $x, y \in \mathcal{C}$, a possible path between them in $L\mathcal{C}$ is something like:

$$\begin{tikzcd} & a & & c & & e & \\ x \arrow[ur] & & b \arrow[ul] \arrow[ur] & & d \arrow[ul] \arrow[ur] & & y \arrow[ul] \end{tikzcd}, \tag{71}$$

where all of the arrows are morphisms in $\mathcal{C}$.

Theorem 4.6: $\pi_1(L\mathcal{C})$? (Bökstedt-Svane)

Let $\mathcal{C}$ be a pointed reversible category. Then there is a surjective homomorphism:

$$\text{GroupComplete}(\text{End}_{\mathcal{C}}(1)) \xrightarrow{\gamma_-} \text{End}_{L\mathcal{C}}(1) = \pi_1(L\mathcal{C}) \tag{72}$$

with kernel generated by the so-called chimera relations:

$$(1 \xrightarrow{f_1} M \xrightarrow{f_2} 1 \xrightarrow{g_1} M \xrightarrow{g_2} 1) \sim (1 \xrightarrow{f_1} M \xrightarrow{g_2} 1 \xrightarrow{g_1} M \xrightarrow{f_2} 1). \tag{73}$$

Proof. First, let us show that this map is surjective. Let $\gamma \in \pi_1(L\mathcal{C})$. In the case of bordisms, this means that every element of π_1 can be expressed in terms of a single closed manifold. From our explanation earlier, γ can be expressed as a zig-zag, say:

$$\gamma = \gamma_{f_1} \cdot \overline{\gamma_{f_2}} \cdots \gamma_{f_{n-1}} \cdot \overline{\gamma_{f_n}} = \begin{tikzcd} & i_1 & & i_3 \arrow[dr] & & \cdots & & i_{n-1} & \\ 1 \arrow[ur, "f_1"] & & i_2 \arrow[ul, "f_2"'] \arrow[ur, "f_3"] & & \cdots & & i_{n-1} \arrow[ul] \arrow[ur, "f_{n-1}"'] & & 1 \arrow[ul, "f_n"] \end{tikzcd}. \tag{74}$$

But then by inserting some identities, we can rewrite this as:

$$\begin{aligned} \gamma &= \gamma_{f_1} \cdot \overline{\gamma_{f_2}} \cdots \gamma_{f_{n-1}} \cdot \overline{\gamma_{f_n}} \\ &= \gamma_{f_1}\left(\gamma_{\overline{f_2}} \cdot \gamma_{f_3} \cdots \gamma_{\overline{f_n}}\right)\overline{\left(\gamma_{\overline{f_2}} \cdot \gamma_{f_3} \cdots \gamma_{\overline{f_n}}\right)} \cdot \gamma_{f_2} \cdot \left(\gamma_{\overline{f_3}} \cdot \gamma_{f_4} \cdots \gamma_{f_n}\right) \cdot \overline{\left(\gamma_{\overline{f_3}} \cdot \gamma_{f_4} \cdots \gamma_{f_n}\right)} \cdot \gamma_{f_3} \cdots \gamma_{\overline{f_n}} \\ &= \gamma_{f_1 \circ \overline{f_2} \circ \cdots \circ \overline{f_n}} \cdot \overline{\gamma_{\overline{f_n} \circ \cdots \circ \overline{f_2} \circ f_2 \circ \cdots \circ f_n}} \cdots \overline{\gamma_{\overline{f_n} \circ f_n}} \\ &= \gamma_{f_1 \circ \overline{f_2} \circ \cdots \circ \overline{f_n}} \cdot \gamma_{\overline{\overline{f_n} \circ \cdots \circ \overline{f_2} \circ f_2 \circ \cdots \circ f_n}} \cdots \gamma_{\overline{\overline{f_n} \circ f_n}} \\ &= \gamma_{(f_1 \circ \overline{f_2} \circ \cdots \circ \overline{f_n}) \circ \overline{(\overline{f_n} \circ \cdots \circ \overline{f_2} \circ f_2 \circ \cdots \circ f_n)} \circ \cdots \circ \overline{(\overline{f_n} \circ f_n)}}. \end{aligned}$$

Thus any based loop γ in the localization of $\mathcal{C}$, i.e a map in $\text{End}_{L\mathcal{C}}(1)$ can be expressed as the image of the giant composition of based loops and their formal inverses in $\mathcal{C}$:

$$\left(f_1 \circ \overline{f_2} \circ \cdots \circ \overline{f_n}\right) \circ \overline{\left(\overline{f_n} \circ \cdots \circ \overline{f_2} \circ f_2 \circ \cdots \circ f_n\right)} \circ \cdots \circ \overline{\left(\overline{f_n} \circ f_n\right)} \in \text{GroupComplete}(\text{End}_{\mathcal{C}}(1)). \tag{75}$$

In other words, **our map γ_- is indeed surjective!**

However, this map clearly has a nontrivial kernel. Indeed, let $f_1, g_1 \in \text{Hom}_{\mathcal{C}}(1, M)$ and $f_2, g_2 \in \text{Hom}_{\mathcal{C}}(M, 1)$ and observe that:

$$\gamma_{f_1} \cdot \gamma_{f_2} = \gamma_{f_1} \cdot (\gamma_{g_2} \cdot \overline{\gamma_{g_2}}) \cdot (\overline{\gamma_{g_1}} \cdot \gamma_{g_1}) \cdot \gamma_{f_2} = \gamma_{f_1 \circ g_2} \cdot \overline{\gamma_{g_2 \circ g_1}} \cdot \gamma_{g_1 \circ f_2}. \tag{76}$$

This means that

$$\gamma_{f_1 \circ f_2 \circ g_1 \circ g_2} = \gamma_{f_1 \circ g_2 \circ g_1 \circ f_2}, \tag{77}$$

which is exactly the **chimera relation!**

We have proved that anything generated by the chimera relation lies inside the kernel of our surjective homomorphism γ_-, but we leave it to the reader to prove that this subsumes the entirety of the kernel. $\square$

Theorem 4.7: The 1-Type of LBord

Now what does all of this mean for the case of interest: $\mathcal{C} = \mathrm{LBord}^X_{n,n-1}$?

First, from our earlier general discussion, we know that $\pi_0(\mathrm{LBord}^X_{n,n-1})$ is just equal to the set of objects in $\mathrm{Bord}^X_{n,n-1}$) modulo the equivalence relation that ($Y^{n-1} \sim Z^{n-1}$ if and only if there exists an X-structured bordism $X^n : Y^{n-1} \to Z^{n-1}$.

But this means exactly that:

$$\boxed{\pi_0(\mathrm{LBord}^X_{n,n-1}) = \Omega^{X(n)}_{n-1},} \tag{78}$$

where $\Omega^{X(n)}_{n-1}$ is just the usual $(n-1)$-dimensional X-structured bordism group.